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EXPLORATORY STATISTICAL ANALYSIS OF PLANET APPROACH-
PHASE GUIDANCE SCHEMES USING RANGE, RANGE-RATE,
AND ANGULAR-RATE MEASUREMENTS

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SUMMARY

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An exploratory analysis of schemes applicable to the guidance and control of the approach to a target planet is presented including digital solutions of statistical probability. Control logic is evaluated on the basis of guidance accuracy and velocity-increment requirements for trajectory correction. Nondimensional results are applicable to any moon or planet within limits of two-body assumptions. The instrumentation system is assumed to produce simultaneous range, range-rate, and angular-rate $\dot{\theta}$ measurements, where $\dot{\theta}$ is the rate of change of the radius vector to the target body with respect to an inertial reference direction. It is assumed the dominant source of error is in the $\dot{\theta}$ measurements. A system of measurement based on radar or radio techniques in conjunction with gyroscopic determination of an inertial reference direction is accurately approximated. Qualitative discussion of possible applicability to other measurement schemes is included.

Results indicate that guidance logic is relatively insensitive to mission energy or to residual errors from midcourse guidance that have a predictable effect on velocity-increment requirements. The guidance scheme is, however, sensitive to the altitude of the target perigee. Low-altitude targets are more accurately attained than those at higher altitude. A large range of $\dot{\theta}$ measurement errors is considered, and use of the radar-gyroscopic system is considered feasible for guidance to atmospheric-drag deceleration accuracies with instruments of current performance expectations.

A strong trend toward heavily "restrained" and "damped" guidance systems is indicated, and a large number of corrections in trajectory following initiation of guidance at long range would be desired. It is indicated that active control with no cutoff is preferred. Control

engines require a range of velocity increments of 50/1 for good control performance, with a minimum acceleration capability of $1/3$ g, or 1 g for drag deceleration approaches.

INTRODUCTION

Many analyses of trajectory and mission requirements on the subjects of orbital, lunar, and interplanetary vehicles have been reported. It is common to consider the launch and orbital guidance requirements in terms of cutoff accuracies (ref. 1), and it is accepted that launch guidance is insufficient to perform manned interplanetary missions without later corrections (ref. 2). The logic and concepts of navigation in three-dimensional, three-body fields and heliocentric orbits have been studied (refs. 2 to 11).

This report is concerned with the terminal phase of a mission outside the planet atmosphere, the approach or approaches to a target moon or planet. Literature on the terminal and reentry phases of a mission includes references 10 to 16. This analysis, however, considers the random sampling of real measurements and a multicorrection (but not continuous) guidance system.

It is customary to expect an accurate approach to a target planet to facilitate the optimum expenditure of retrothrust or to permit the use of atmospheric-drag deceleration, or a combination of both. As a result of the relative inability to measure precise trajectory parameters at large distances in space, it logically follows that the burden of guidance and trim falls on the later phases of a mission when accurate determination becomes feasible. Though no physical justification should be implied, the mathematical complexity of three-body problems makes it expedient to consider only the approach phases of a mission where two-body approximations can be used.

An exploratory analysis of this problem using electronic digital computer techniques and a modified Monte Carlo method of calculation has been performed at NASA Lewis Research Center and is described herein. The principal interests in this study are evaluation of velocity-increment requirements and guidance philosophy for a system with significant errors in the determination of trajectory angular rate, or the rate of rotation of the vehicle around the target body with respect to inertial space.

A direct reading in trajectory angular rate can be hypothesized to result from measurement of the rate of change of the angle between the direction of the range-sensing instruments and an inertially stabilized platform. Thus, the analysis may be considered an evaluation of possible gyroscopic components of a guidance system. The range and range-rate measurements could result from radar, radio, and sometimes optical or

infrared instrumentation. Though no exclusion of ground-based components is implied, the discussion is applicable to completely vehicle-contained systems acting on passive planets. It is shown that range and range-rate measurement errors have relatively insignificant effect if radio or radar instrumentation is used and in some cases if optical or infrared instrumentation is used. The discussion is slanted toward the use of gyroscopes to determine an inertial reference direction, but other schemes (such as the use of accelerometers) may be considered if the errors permit valid analysis with assumptions used herein.

It is desired to control the vehicle so that its minimum altitude occurs as the trajectory is tangent to an arbitrary target sphere. In other words, the target of guidance is an arbitrary perigee range. To obtain more practical assessments of velocity-increment requirements, the direction of rotation is specified, but the point of tangency is not specified, and a two-dimensional polar representation is therefore sufficient for analysis. It is assumed that thrust application is impulsive and duplicates identically the demands of the guidance system.

The extensive and complex fields of smoothing techniques, reliability, computer equipment, and measuring instruments are not considered herein. In contrast, the performance of complete systems is postulated and the guidance logic and philosophy are evaluated relative to the assumed systems.

Results are analyzed primarily on the basis of the probability of error in final perigee and the probability of exceeding total velocity-increment capabilities. The expected frequency and magnitude of individual velocity increments are obtained. Results are nondimensionalized and are valid for all planets, subject, of course, to the previous two-body assumption.

Since consideration of guidance equipment available, weights, engines, reliability, mission, and many other factors would enter into final arbitration, the optimization of results is far beyond the scope of this study. Instead, the effects of some major parameters have been investigated, including the following:

- (1) Effects of error assumptions
- (2) Trajectory considerations
- (3) Guidance considerations

Finally, a sample calculation and a discussion of the computing methods are included in appendixes.

SYMBOLS

E	nondimensional total energy per unit mass (eq. (1))
$\mathcal{E}$	total energy per unit mass, $\text{g-cm}^2/\text{sec}^2\text{-g}$
err^P	distribution of perigee errors
$\text{err}^{\dot{\Theta}}$	distribution of measurement errors
F	function
f	frequency
G	universal gravitational constant, $\text{dyne-cm}^2/\text{g}^2$
H	nondimensional angular momentum per unit mass (eq. (2))
h	angular momentum per unit mass, $\text{g-cm}^2/\text{g}$
K	restraint factor in guidance logic (eq. (17))
M	mass of target body, g
N	number
P or R_p	nondimensional perigee range, r_p/r_o
p	probability (appendix D)
R	nondimensional range, r/r_o
r	range, cm
t	time, sec
V	nondimensional velocity, v/v_e
ΔV	nondimensional velocity increment due to thrust, $\Delta v/v_e$ (eq. (5) or (6))
v	velocity, cm/sec
α	angle from V_1 to ΔV , radians (eq. (5) or (6), fig. 1(b))
β	angle from V_a to ΔV , radians (appendix A)
γ	angle from the normal to R to V, radians (eq. (2), fig. 1(a))
ϵ	eccentricity

$\dot{\Theta}$	nondimensional trajectory angular rate, $d\Theta/d\tau$
$\dot{\Theta}_{cal}$	calibration expected measurement error defining dead band in logic (eq. (16))
σ	standard error
τ	nondimensional time, tr_o/v_e
τ_{go}	nondimensional time to perigee
ϕ	angle from R to P, radians (eq. (2), fig. 1(a))
ω	apparent half-angle, radians (appendix C)

Subscripts:

a	actual
av	average
e	escape
err	error
i	indicated by measurements with errors
id	ideal
max	maximum
min	minimum
o	surface of body (planet)
p	perigee
tar	target
tot	total
1	before thrust application
2	after thrust application

Superscript:

$(\dot{})$	rate of change with nondimensional time, $d()/d\tau$
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Presuperscript:

N number of increments

ANALYSIS

This exploratory analysis into the schemes and philosophy applicable to guidance and control of the approach to a target planet considers navigation systems having significant measurement errors. The arbitrary but realistic limitation of the "approach" to within the central force field of the target planet, but above its atmosphere, leads logically to the use of two-body trajectories as closely representing the physical situation of interest.

The following analysis will first illustrate the relations and non-dimensional forms of the conic trajectories, leading into a discussion of the errors and the significance of errors for measurement schemes. Some basic trends and relations are demonstrated. The control functions and logic are then hypothesized and developed.

Finally, a description of the statistical mathematics used in digital computation is presented, and the general method of statistical evaluation is briefly described.

Trajectory Relations

With the assumption of two-body relations, the governing equations can be expressed as laws of conservation of energy and angular momentum (refs. 17 and 18). With the notation of figure 1(a),

$$\mathcal{E} = \frac{v^2}{2} - \frac{GM}{r} \quad (\text{energy per unit mass})$$

$$h = vr \cos \gamma \quad (\text{angular momentum per unit mass})$$

A convenient nondimensional form for the problem considered herein is obtained using the parabolic escape velocity at the surface of the target planet as a reference:

$$\mathcal{E}_e \equiv 0 = \frac{v_e^2}{2} - \frac{GM}{r_0}$$

Defining now,

$$R \equiv \frac{r}{r_0}, \quad V \equiv \frac{v}{v_e}, \quad \text{etc.}$$

so that R is measured in radii and V in escape velocities (from the surface). Working forms become

$$E \equiv \frac{c}{v_e^2/2} = V^2 - \frac{1}{R} \quad (1)$$

$$H \equiv \frac{h}{v_e r_0} = R(V \cos \gamma) = R^2 \dot{\Theta} \quad (2)$$

where $\dot{\Theta} = d\phi/d\tau = d\Theta/d\tau$ with units of radians, or r_0/v_e radians per time.

This referencing system permits a general analysis valid for any planet or moon.

Target perigee. - The target of the hypothetical missions considered has been chosen as the minimum range R of the conic trajectory, the perigee, $R_p \equiv P$. Specifically, it is desired to guide the vehicle to an arbitrary minimum altitude at which horizontal velocity will occur. However, the orientation of the plane of motion is not defined, but it is of interest to guide the vehicle to the desired direction of rotation around the target planet.

For $R = P$, $\gamma = 0$, equations (1) and (2) become

$$H = V_p P \quad \text{and} \quad E = V_p^2 - \frac{1}{P}$$

Therefore,

$$E = \frac{H^2}{P^2} - \frac{1}{P} \quad (3)$$

and the perigee range is then

$$\left. \begin{aligned} P &= \frac{1}{2E} (\pm \sqrt{1 + 4EH^2} - 1) & E \neq 0 \\ P &= H^2 & (\text{from eq. (3)}) \quad E = 0 \end{aligned} \right\} \quad (4)$$

The vehicle is expected to have thrust capability to guide it to the target perigee P_{tar} .

Corrective thrust. - Throughout this analysis, velocity increments ΔV caused by thrust will be considered representative of propellant

expenditure. In nondimensional form ΔV is measured in escape velocities.

The ΔV used for guidance is assumed to be impulsive in effect; that is, ΔV is applied over relatively short periods of time compared with trajectory times. It is further assumed that ΔV is applied at constant inertial direction in the plane of vehicle motion. Using the notation of figure 1(b) with the law of cosines:

$$\left. \begin{aligned} \Delta V^2 &= V_1^2 + V_2^2 - 2V_1V_2 \cos(r_2 - r_1) \\ \alpha &= \cos^{-1} \frac{V_1^2 - V_2^2 + \Delta V^2}{2V_1\Delta V} \end{aligned} \right\} \quad (5)$$

where α is the direction of ΔV application relative to the initial velocity V_1 .

It is a prime objective to minimize total ΔV for the approach. Logically, then, it is desired to minimize equations (5) to determine the direction α at which minimum individual ΔV increments will occur. An involved pair of working equations in two unknowns is easily obtained but requires an iterative solution. Because the statistical problems anticipated require repeated calculation of ΔV , the use of optimum values becomes tedious.

A reasonable approximation of the optimum solution for ΔV is obtained if $V_2 = V_1$ (for small values of $r_2 - r_1$). The degree of approximation is shown in reference 16. With impulsive ΔV and $V_2 = V_1$, the energy becomes constant ($E_2 = E_1$), and

$$\left. \begin{aligned} \Delta V &= 2V_1 \sin\left(\frac{r_2 - r_1}{2}\right) \\ \alpha &= \frac{\pi}{2} - \left(\frac{r_2 - r_1}{2}\right) \end{aligned} \right\} \quad (6)$$

where r_2 is readily obtained from the desired perigee after correction P_2 along with equations (1), (2), and (3). Thus, ΔV and α are obtained by closed solution.

In practice, however, ΔV is calculated from measured parameters including errors. The actual trajectory after correction must be calculated using the ΔV indicated and the actual trajectory before correction. This exercise in geometry is shown with a sample calculation in appendix A. Of greater interest here is the significance of measurement errors.

Analysis of Measurement Errors

In order to concentrate the emphasis of this analysis on the logic and philosophy of guidance techniques, it is assumed that all preliminary calculations and smoothing are performed before data input to the guidance computer. The details of measurement, computing equipment, and smoothing methods are not considered.

The guidance scheme considered makes use of simultaneous information:

- (1) Range, R
- (2) Range rate, $\dot{R}$
- (3) Trajectory angular rate, $\dot{\Theta}$

It is noteworthy, however, that the three readings could be obtained by continuous measurement with a radar-gyroscopic combination of instruments, and that finite difference approximations need not be used.

Using $V^2 = \dot{R}^2 + (R\dot{\Theta})^2$ and equations (1) and (2),

$$\left. \begin{aligned} E &= \dot{R}^2 + (R\dot{\Theta})^2 - \frac{1}{R} \\ H &= R^2\dot{\Theta} \end{aligned} \right\} \quad (7)$$

The trajectory parameters are readily obtained with only one set of measured data.

To determine the effect of measurement errors on the perigee range, note that

$$E = \frac{H^2}{P^2} - \frac{1}{P} \quad (3)$$

Differentiating and solving for dP ,

$$dP = \frac{2H \, dH - P^2 \, dE}{2PE + 1} \quad (8)$$

$$dP = \frac{\partial P}{\partial R} dR + \frac{\partial P}{\partial \dot{R}} d\dot{R} + \frac{\partial P}{\partial \dot{\Theta}} d\dot{\Theta}$$

Finally,

$$\left. \begin{aligned} \frac{\partial P}{\partial R} &= \frac{1}{2PE + 1} \left[\left(2 - \frac{P^2}{R^2} \right) \left(\frac{2H^2}{R} \right) - \frac{P^2}{R^2} \right] \\ \frac{\partial P}{\partial \dot{R}} &= \frac{1}{2PE + 1} (-2P^2 \dot{R}) \\ \frac{\partial P}{\partial \dot{\Theta}} &= \frac{1}{2PE + 1} [(R^2 - P^2) 2H] \end{aligned} \right\} \quad (9)$$

Equation (9) will be considered in detail later. For simplicity, consider the special case of a parabolic trajectory grazing the surface of the target planet. With $E = 0$ and $P = 1$, equations (9) reduce to

$$\left. \begin{aligned} \frac{\partial P}{\partial R} &= \frac{2}{R} \left(2 - \frac{1}{R^2} \right) - \frac{1}{R^2} \\ \frac{\partial P}{\partial \dot{R}} &= -2 \frac{\sqrt{R-1}}{R} \\ \frac{\partial P}{\partial \dot{\Theta}} &= 2(R^2 - 1) \end{aligned} \right\} \quad (10)$$

Taking now the limiting case of large range, the error coefficients approach the limits

$$R \gg 1 \left\{ \begin{aligned} \frac{\partial P}{\partial R} &\approx \frac{4}{R} \rightarrow 0 \\ \frac{\partial P}{\partial \dot{R}} &\approx \frac{-2}{\sqrt{R}} \rightarrow 0 \\ \frac{\partial P}{\partial \dot{\Theta}} &\approx 2R^2 \rightarrow 2R^2 \end{aligned} \right\} \quad (11)$$

so that at large range the error resulting from angular-rate measurement errors becomes proportional to R^2 , while the other sources have insignificant effect. The range of error coefficients is illustrated in figure 2 for the parabolic trajectory with $P = 1$.

The error coefficient of the gyroscopic measurement, $\dot{\Theta}$, is 10 times that of either range or range-rate measurements at a range of $2\frac{1}{2}$ radii (10,000 statute miles), but is a full 5 decades greater at a range of 100 radii (400,000 miles). Parenthetical values here, and henceforth,

indicate rough equivalents for a terrestrial approach. Note also that the error coefficient for the $\dot{\Theta}$ measurement varies more than 4 decades over the range from 1 to 100 radii.

The extreme sensitivity in perigee determination due to $\dot{\Theta}$ measurement is ~~predominantly~~ an inherent difficulty in determining the horizontal component of velocity, as mentioned in reference 6. Also, it should not be inferred that the choice of a gyroscopic measurement leads to the sensitivity, since, on the contrary, the significant fact is that gyroscopic instruments measure sensitive parameters. This is further illustrated in appendix B, where errors in perigee due to R and $\dot{R}$ readings alone are considered. It is shown that the roughly 4-decade variation in sensitivity with range occurs (within a factor of 2) for the $R, \dot{R}$ scheme also.

In consistent units, the numerical size of R or $\dot{R}$ measurement error by radar or radio instrumentation is expected to be 1 or probably 2 decades smaller than that in gyroscopic measurement of $\dot{\Theta}$. Thus, errors in perigee determination due to $\dot{\Theta}$ errors are dominant over the entire range of interest. It is thus a practical simplification to assume $\dot{\Theta}$ the sole source of error.

If range and range rate are to be determined by optical or infrared techniques, a general conclusion concerning error significance is not available. It can only be indicated that some combinations of instrumentation lead to the error in $\dot{\Theta}$ as the dominant source of perigee error. For these, this analysis is applicable. An evaluation of this measurement scheme is included in appendix C.

The perigee error due to measurement of $\dot{\Theta}$ is also a function of the trajectory - in particular, the perigee itself. Using equation (3) to eliminate angular momentum from equations (9),

$$\left. \begin{aligned} \frac{\partial P}{\partial \dot{\Theta}} &= \frac{\pm 2}{2PE + 1} \left[\sqrt{P^2 E + P} (R^2 - P^2) \right] & E \neq 0 \\ \frac{\partial P}{\partial \dot{\Theta}} &= \pm 2\sqrt{P} (R^2 - P^2) & E = 0 \end{aligned} \right\} \quad (12)$$

The $\dot{\Theta}$ error coefficients for parabolic trajectories with a series of perigee values are shown in figure 3, again as a function of range. At a given range the error coefficient is larger if the perigee is larger, except that, as the range decreases toward the perigee, the coefficient decreases rapidly, vanishing at $R = P$. The envelope of maximum error coefficient at any R is considered next. In passing, however, note that a vehicle on a trajectory with a given perigee is unlikely to coast past that perigee and into the outward leg of the trajectory because of

errors in $\dot{\theta}$ measurement, since, as $R \rightarrow P$, the error sensitivity tends to vanish.

The maximum error coefficient for the $\dot{\theta}$ measurement at any R is a useful measure of expected P errors. To this end, equation (12) is maximized with respect to perigee P :

$$\frac{\partial \left(\frac{\partial P}{\partial \dot{\theta}} \right)}{\partial P} = 0 = \frac{\pm 2 \sqrt{P^2 E + P(R^2 - P^2)}}{2PE + 1} \left[\frac{2PE + 1}{2P(PE + 1)} - \frac{2E}{2PE + 1} - \frac{2P}{R^2 - P^2} \right] \quad (13)$$

After simplification,

$$\left. \begin{aligned} \frac{R^2}{P^2} - 1 &= 4(PE + 1)(2PE + 1) & E \neq 0 \\ \frac{R^2}{P^2} &= 5 & E = 0 \end{aligned} \right\} \quad (14)$$

so that, for any trajectory given by energy E and perigee P , the curve of maximum error is determined. The maximum error coefficient is (after simplification and substitution into eqs. (12))

$$\left. \begin{aligned} \frac{\partial P}{\partial \dot{\theta}} &= \pm 8P^{5/2}(PE + 1)^{3/2} & E \neq 0 \\ \frac{\partial P}{\partial \dot{\theta}} &= \pm 8P^{5/2} = \pm \frac{8R^{5/2}}{5^{5/4}} \approx \pm 1.07R^{5/2} & E = 0 \end{aligned} \right\} \quad (15)$$

where the nonzero energy relations are more readily used in the parametric form (eqs. (14) and (15)) than solved for as $F(E, R)$.

The maximum error coefficient at a range R is shown in figure 4 as a function of range for several values of energy. The log-log representation of the zero-energy error forms a straight line with R having a slope of $5/2$. The higher-energy (hyperbolic) maximum error coefficients are smaller than those of the parabolic trajectory. The energy variation from parabolic to one-unit hyperbolic represents a substantial change; for example, a launch from Earth to $E = 1$ would require almost 18,000 feet per second over escape velocity. For negative energy (elliptic) the maximum error coefficients are generally higher than the parabolic, but indicate a maximum value and a decrease as the apogee is approached.

It now becomes feasible to estimate the uncertainties in perigee due to measurement errors. Arbitrarily taking an error of 10^{-3} radian (23-sec arc per minute as a drift rate or 2-min arc in each of two readings with a 10-minute interval). Actual perigee errors can be estimated from figure 4 using the assumed measurement error:

$$dP = \frac{\partial P}{\partial \dot{\theta}} d\dot{\theta} = 10^{-3} \frac{\partial P}{\partial \dot{\theta}}$$

At a range of 2 radii the error in perigee varies from about 3 to 7×10^{-3} radius (12 to 28 miles). (Parenthetical values again refer to an Earth approach.)

Since linearization is implied in using error coefficients, it is not reasonable to indicate a perigee error of $100,000 \times 10^{-3} = 100$ radii at a range of 100 radii; however, the error is large. Inversely, to correct vehicles at 100 radii to an accuracy of the order 10^{-3} radius, instrument errors must be of the order of 10^{-7} or 10^{-8} , where current expectation is of the order 10^{-3} to 10^{-4} or at least 3 decades poorer (ref. 6 or 7).

Two important results of the preceding discussion are: (1) The use of excess-energy trajectories does not complicate the guidance problem; actually, smaller errors in perigee determination would occur. (2) Reduced error coefficients at small R make active control to small R advantageous, and the use of "cutoff" at large R is discouraged.

Control Functions

As previously indicated, the primary objectives of control and guidance are (1) to direct the vehicle to the desired direction of rotation and perigee altitude, and (2) to do so with the minimum expenditure of thrust. The problem analyzed here differs in concept from the problems of classical control because it is seldom of principal importance to minimize control power expenditure.

Reiterating the scope of the problem as previously defined, consideration is given to the two-body trajectory modification by perfectly executed impulses in thrust acting on information including errors. The mechanics of instrumentation, smoothing, and preliminary calculation are not considered. In addition, the assumption of a single significant error (that in $\dot{\theta}$) is made.

With minimum control action (thrust) as a prime objective, the use of a dead band to reduce oscillation of the path around the desired trajectory is mandatory. The inclination of the target perigee is undefined; therefore, it is possible to represent the course of a vehicle to

$$P = F(R, \dot{R}, \dot{\Theta})$$

or, since $\dot{\Theta}_i$ is the measured parameter including errors,

$$P = F(R, \dot{R}, \dot{\Theta}_a, \dot{\Theta}_{err})$$

and considering the actual and known values as constants, logic can be based on the uncertainty in measurement of angular rate alone:

$$errP = F(err\dot{\Theta})$$

where the err presubscript differs from a postsubscript in that it represents a statistical probability of errors. The preceding relation is thus interpreted as "the probable error in perigee is a function of the probable error in $\dot{\Theta}$." (The particular function would depend on the constants E , $\dot{R}$, R , etc.)

The dead band is not easily related to the error in perigee because of the complex relations (eqs. (8) to (15)) that must be treated. It is not unreasonable to expect the instruments used in measurement to be accurately calibrated before use. It is then natural to use this calibration data to aid in formulating guidance logic. The dead band is defined in terms of the calibration of the instrumentation; it is treated as a constant and is considered a "confidence" placed in the measured values of $\dot{\Theta}$. Dead band is symbolized by $\dot{\Theta}_{cal}$.

The error in trajectory as determined by measurements is linked to the use of corrective ΔV by the logic

$$|\dot{\Theta}_i - \dot{\Theta}_{tar}| \geq \dot{\Theta}_{cal} \quad (16)$$

so that ΔV is used if the indicated error is greater than the measurement error expected from knowledge of instrument calibration. In other words, unless the error in trajectory as indicated by $|\dot{\Theta}_i - \dot{\Theta}_{tar}|$ is greater than the errors known to be generated by the instruments, a coasting trajectory is continued without using ΔV .

The dead band $\dot{\Theta}_{cal}$ is used to prevent oscillation within the region of "no confidence." Outside the dead band, errors can occur such that the corrective action would cause overshoot of the desired trajectory with the eventual necessity of using ΔV to cancel the overshoot. It remains a question, therefore, whether all of the "indicated" error should be corrected, since part of it is of uncertain origin and could be caused by instrument errors.

After the use of corrective ΔV , the indicated perigee error is

$$|\dot{\Theta}_i - \dot{\Theta}_{tar}|_2 = K \dot{\Theta}_{cal} \quad (17)$$

where K is termed the "restraint factor" and is the fractional part of the indicated error of uncertain origin left uncorrected. Guidance using a $K = 0$ attempts to correct all of the indicated error, while $K = 1$ guidance would correct only the part of the error considered "real."

The logical determination of good or optimum values of $\dot{\Theta}_{cal}$ and K requires the complexity of a statistical solution because of the interaction of trajectory and guidance logic.

ΔV Limitations. - The control logic also must function to maintain the demands on ΔV within the economic and practical limits of the engines. Too-small corrections may in some circumstances result in prohibitive startup or shutdown losses, while too-large impulses might cause overheating or oversized engines.

The "control" is supplied with maximum and minimum values of ΔV . If demand is indicated greater than the maximum ΔV_{max} , the control executes only the maximum allowable. If, on the other hand, demand ΔV is less than the minimum ΔV_{min} , control thrust is omitted.

Sampling rate. - The final function of control is to govern the rate of sampling data and executing corrective actions. The sequence of ΔV corrections between the initial and final (or cutoff) points is of major importance in guidance. The optimization of this sequence is dependent on a large number of factors, including the rate at which accurate data can be obtained. Thus, a time-variant sequence is used in this analysis. The solution of the conic equations for time is indicated in appendix A. By using the time at the perigee as a limit in integration, the time to go to the perigee is obtained.

Two major considerations prevent the execution of optimization with this analysis: (1) Factors related to smoothing techniques, measurement equipment, and thrust devices must be assumed in detail, and (2) the statistical analysis did not possess the "fineness" to consider high rates of sampling. Arbitrary sequences are used herein, with an increment size proportional to τ_{go} as the primary variation; but, since the proportional sequence leads to vanishing $\Delta\tau_{go}$ as $\tau_{go} \rightarrow 0$, a minimum limit is placed on the increment size. The result is a sequence with large $\Delta\tau_{go}$ steps initially, decreasing to smaller steps, finally followed by a series of constant intervals in time.

The time of the initial correction is dependent on the radar range available and on the breakdown of the two-body assumptions as range is increased. The choice then would depend on the radar equipment used and on the planet (or moon) approached. The cutoff or final correction has been discussed previously, with the conclusion that cutoff should be as late in time, or as close to the perigee, as possible.

Method of Statistical Analysis

The method of computation is described here in general terms to illustrate the treatment of the "family" of vehicles. Details, special cases, and equations are omitted where possible, but are considered in the sample calculations of appendixes A and D.

A standard Monte Carlo solution would consider the guidance logic for a single vehicle (case) during an approach, using the generation of random errors. The statistics of guidance would then be developed from a number of "cases." High probability or average results are easily represented with relatively small numbers of cases. However, unlikely events are represented with accuracy only by very large numbers of cases. In contrast to conventional Monte Carlo methods, it is possible to consider "all" cases by approximating the errors by frequency polygons with finite increments. The solution would result directly in the statistics of probability as soon as all combinations of variables were considered. For the problem under study any "result" might be represented in the form:

$$\text{Result} = F(\text{err}E_i, \text{err}P_i, \text{err}^{\odot}_1, \text{err}^{\odot}_2, \dots, \text{err}^{\odot}_n)$$

where n is here the number of sampling points, so that any result is a function of initial trajectory and errors in measurement at each point of control action. The number N of incremental cases contributing to the result is then

$$N_{\text{tot}} \sim \binom{N}{\text{err}E} \binom{N}{\text{err}P} (\text{err}^{\odot})^n$$

so that, if each frequency polygon could be accurately represented by $N = 10$ increments, and if only $n = 20$ sampling points were used, 10^{22} cases must be treated. This number of calculations is obviously excessive.

To reduce the number of cases treated, it is possible to act on the initial set of tabulated data where

$$\text{Result}_1 = F(\text{err}E_i, \text{err}P_i, \text{err}^{\odot}_1)$$

and to regroup the results into a new set of tabulated data. It is of importance that the identity of particular vehicles is lost in the regrouping process. However, if sufficient detail (fineness) is maintained in the tabulating process, the physical problem is accurately represented. The number of cases to be considered as the calculation and regrouping is repeated n times is:

$$N_{\text{tot}} \sim (\text{err}^N_E)(\text{err}^N_P)(\text{err}^{\dot{N}\theta})(n)$$

so that, with $N = 10$ and $n = 20$ as before, the number of cases is 2×10^4 instead of 10^{22} .

It is obviously concluded that contraction methods (regrouping) are needed, and that the gains available are substantial. However, difficulties in interpretation of results are introduced that are inherent to the use of contraction. Any attempt to eliminate these difficulties by rigorous methods leads directly to the elimination of contraction and the return to the huge numbers of cases. Approximate methods of analysis will be developed after one additional simplification.

The family of vehicles to be guided can without complication be defined as having like values of energy initially. It was indicated in deriving relations for ΔV (eq. (6)) that energy changes due to control ΔV should be zero. Assume now that energy change due to the incorrect application of ΔV resulting from measurement errors is insignificant. (Preliminary results indicate that the assumption is physically realistic to an amazing degree when guidance is good; e.g., the extreme cases accumulate a deviation of about 10^{-4} or possibly 2×10^{-3} for a 20-sample approach. This is equivalent to a velocity change of 0.005 and less in ΔV for control purposes.) With trajectory energy a constant, the number of cases to be treated is:

$$N_{\text{tot}} \sim (\text{err}^N_P)(\text{err}^{\dot{N}\theta})(n)$$

With the same estimates as before, 2000 points of logic would be required. In the calculations of this analysis, typical numbers of perigee distribution increments ranged from 80 down to 20, as range decreased, and the number of sampling points was 23. For a large part of the data, 5 increments were used to represent $\text{err}^{\dot{N}\theta}$; for the rest, 10 or 11 were used.

Thus, $N_{\text{tot}} = 7500$ or $15,000$, so that $1\frac{1}{2}$ to 3 hours of computing time is used with an IBM 653 computer.

The statistical evaluation of this analysis uses only the concepts of addition and multiplication of frequency polygons with the assumption that the data are uncorrelated. The details and mechanics of the algebra

are treated in appendix D and can be found in most mathematical statistics texts, such as references 19 and 20.

Three indicators of ΔV requirements are used in the results:

- (1) The average expenditure of ΔV for the whole family of vehicles
- (2) The frequency of application of individual ΔV control actions, as a measure of engine size requirements
- (3) The frequency of total ΔV application during the approach, on the assumption that successive applications are uncorrelated.

It is important to recall that none of the measures of ΔV expenditure permit identification with particular vehicles. For this and other reasons noted in detail in appendix D, the use of a combined (accuracy with ΔV) probability with these data is discouraged.

RESULTS AND DISCUSSION

In order to maintain as high a degree of generality as is consistent with necessary simplifications, most of the assumed parameters are individually varied, and the resulting effect on guidance considerations is presented. However, the multiplicity of possible combinations makes it expedient to assign values arbitrarily to all variables except those of immediate concern. As will be reiterated, the choice of optimum solutions must await final arbitration with factors beyond the scope of this investigation.

An optimum-type solution is presented, however, to illustrate the preferred logic of guidance and the numerical results of the analysis within an arbitrary framework of trajectory parameters, system errors, and computational limitations. This "conditional optimum" solution is then compromised by reducing the sampling rate to produce a reference solution practical for use as a base-line in parametric studies. A compromise is prompted by computing time considerations. Since all of the assumed terms and parameters (except the sampling rate) are alike, they are presented for the conditional optimum and reference solutions simultaneously.

Assumed Values Used in Conditional Optimum and Reference Solutions

As a foundation for presentation methods, and as an example of assumed values common through the investigation, the inputs of the conditional optimum and reference solutions are presented in detail.

The measurement error distribution $\text{err}^{\dot{\theta}}$ is assumed to be rectangular in shape with maximum errors of $\pm 10^{-3}$ radian per unit dimensionless time. The rectangular shape signifies equal probability of error anywhere within the limits. For these solutions the distribution is broken into five equal intervals, each representing 20 percent of the samples.

The assumed initial distribution of errors in trajectory err^P is also rectangular in shape, and the maximum error is ± 10 radii. Errors represent a difference from the target perigee, and negative values of perigee ($dP + P_{\text{tar}} < 0$) indicate the opposite direction of rotation. The distribution is broken into 40 increments each for positive and negative errors, in decades to cover the scope of perigee errors from 10^{-3} to 10. The use of linear increments within each decade results in effectively larger groups in the 1 to 2 part than in the 9 to 10 of each decade. Thus, a somewhat coarse treatment results. (Note: A linear set of floating increments that at any single time covers only the needed range of dP is recommended for future effort.)

The assumed error distributions and the remainder of the assumed parameters are summarized in table I. The values are grouped into the general classes of control functions, trajectory parameters, and assumed error distributions, the last group as just discussed.

The first of the control functions is the dead-band size $\dot{\theta}_{\text{cal}}$, assumed to be 10^{-3} radian per unit dimensionless time. The dead band is thus equal to the largest error in the assumed distribution. The restraint factor K is assumed to be 1.0. These two control functions, together with the assumed errors $\text{err}^{\dot{\theta}}$, should completely prevent overshoot and oscillation of the trajectory as a result of thrust application.

The time of the first corrective action, $\tau_{\text{go},i}$, is 676.8 units, equivalent to a range of 100 radii for a vehicle on the ideal trajectory to the target. Successive sampling points occur as τ_{go} is reduced by increment $\Delta\tau_{\text{go}}$ equal to 5 and 20 percent of τ_{go} until the increment becomes smaller than the minimum $\Delta\tau_{\text{go},\text{min}}$, when constant intervals of 0.75 and 3.0 units are substituted for the conditional optimum and reference, respectively. This sequence results in 95 and 23 sampling points with the last at about 0.2 unit, since no prior cutoff is used in either case.

No maximum limit is placed on individual ΔV , but a minimum value of 0.001 escape velocity is used to reduce the number of very small corrections in the calculation. It is to be shown later that this has no significant effect on results.

The final group of parameters defines the desired trajectory in terms of energy and target perigee. Assumed are parabolic trajectories with a target just above the surface of the planet; $P_{tar} = 1.02$ radii (80-mile altitude).

Results of conditional optimum solution. - The accuracy with which the assumed guidance controlled the vehicles to the target perigee is shown in figure 5(a) as a bar chart of the final distribution around the target. Guidance has reduced the initial perigee errors of ± 10 radii to ± 0.004 radius (16 miles). The same results are shown in figure 5(b) to indicate the probability of hitting a target of arbitrary size. The number of vehicles within an absolute miss distance $|dP|$ is plotted as a function of miss distance. As anticipated, for an allowable miss of 0.004 radius, the probability of success in hitting the target is 100 percent.

The number and size of individual ΔV used for control of the approach are shown in figure 5(c). The percentage of ΔV is relative to the total number of logic points where ΔV could be used. As indicated, 86.5 percent of the time, no ΔV is used. The maximum individual ΔV is in the region 0.08 to 0.09 escape velocity and is required 4×10^{-8} percent of the time. The smaller ΔV (shown in insert) are used more frequently, with the smallest allowable being required 5 percent of the time. The maximum and minimum values represent 3200 and 35 feet per second approaching Earth. It is later to be shown that the maximum can be reduced without cost to 0.05 and possibly 0.04. Also indicated in the figure is the average expenditure of ΔV , 0.0635 escape velocity (2200 ft/sec).

In addition to the frequencies and sizes of ΔV required, it is of interest to consider the expected total expenditure of ΔV for the approach. A bar chart of the expected total ΔV for the guidance of the conditional optimum solution is shown in figure 5(d). Total ΔV expenditures from as little as essentially zero to about $0.12 V_e$ are expected; and, for as high as $0.16 V_e$, possibility exists but with no significant probability. The mode, or most probable point, of the distribution is 0.03 to 0.04, which occurs with 18.7 percent of the vehicles. The average expenditure is 0.0635 escape velocity.

The total use of ΔV during the approach in integrated form is shown in figure 5(e). This type of presentation yields two equally valid methods of interpretation:

- (1) To cause an arbitrary number of vehicles to complete the approach, the ΔV requirement is determined, or
- (2) For an arbitrary ΔV capability available, the number of vehicles successfully completing the approach is determined.

The "ideal" ΔV requirement is the ΔV needed to correct the assumed errors in perigee at a range of 100 radii. The cost, in terms of ΔV requirements, of guidance with errors is about twice the ideal cost. The average ΔV expenditure is again noted, and is sufficient to perform the approach 85 percent of the time. Doubling the ΔV available to, say, 0.13 (4500 ft/sec) would raise the number of vehicles successful in guiding without running short of ΔV to about 99.8 percent.

The largest ΔV required for the last control action imposes the only significant limit on the acceleration capability of the vehicle. The total time remaining to the perigee is about 0.2 unit (2 min), and the ΔV required may be 0.03 to 0.04 (1050 to 1400 ft/sec). Actually, not all of this time is available with a drag deceleration mission, since the atmosphere is encountered some distance before the perigee.

Violating at this point the assumption of impulsive ΔV , the minimum acceleration capability a of the vehicle can be roughly estimated as

$$a = \frac{dV}{dt} \sim \frac{\Delta V}{\Delta t} \sim \frac{0.04}{0.2} = 0.2 \quad (a \sim 12 \text{ ft/sec}^2)$$

or, assuming $1/3$ of the last interval available,

$$a \sim 0.6 \quad (a \sim 36 \text{ ft/sec}^2 \sim 1g)$$

Thus, the vehicle will require fairly large control thrust capability and may often be using the main propulsion devices for control purposes. Note also that ΔV is applied roughly perpendicular to the velocity of the vehicle.

Results of reference solution. - The results of the conditional optimum solution were illustrated to show some numerical values anticipated with good guidance logic. The computing time required with the high sampling rate reached limits of practicality, however, and a reference solution is presented that expediently permitted reduction in run time by factors of 4 or better. Subsequent effects of individual parameters will be referred only to the reference solution. The two solutions are not compared here, but results are shown later as trends with variation of sampling rate as a guidance factor.

The accuracy of control to the target perigee for the reference solution is shown as a bar chart in figure 6(a). Guidance has reduced positive target miss distances to 0.004 radius as with the optimum guidance, but negative errors as large as 0.02 radius (80 miles) are indicated, the two blocks from -0.009 to -0.020 representing almost 25 percent of the vehicles.

Because sampling points are determined from arbitrary time increments (equal for all trajectories) and are now quite large, vehicles with smaller perigee than the target coast to a range R less than P_{tar} between the final two corrections (sampling points). Once this occurs, no single thrust impulse exists to correct to $P_{tar} > R$. The difficulty illustrated is tolerated primarily because the anticipated comparisons of trends are not significantly hampered.

The probability of success in hitting an arbitrary target is shown in figure 6(b). As expected, the miss distance for 100-percent success is 0.02 radius.

The use of individual velocity increments for control purposes is shown in figure 6(c), and the average or total expenditure of $0.092 V_e$ is indicated. With ΔV arbitrarily limited to above $0.001 V_e$ (35 ft/sec), the demand covers a scope of 100/1 with 0.1 ΔV required by only about 10^{-3} percent of the logic points. Again, the demand for smaller corrective actions is dominant, and corrective action is omitted at 67 percent of the logic points.

The total ΔV requirements are shown in figure 6(d) as the probability of success with an arbitrary availability of ΔV . The average requirement is sufficient to perform the approach 65 percent of the time, or for 65 percent of the vehicles; and double the average ΔV ($0.18 V_e$) is sufficient 98 percent of the time.

Significance of Error Assumptions

Before using results to indicate the effects of control functions, it is useful to obtain estimates of the significance of assumed errors and of the approximations in representing the error distributions. Considering first the errors in angular-rate measurement $err_{\dot{\theta}}$, the size, shape, and number of intervals used in digital approximation will be investigated.

Number of intervals in approximating $err_{\dot{\theta}}$. - The probability of vehicles' hitting a target of arbitrary size is shown in figure 7(a) for results computed with 10 and 5 (reference) increments used to approximate the $err_{\dot{\theta}}$. As will be the general procedure herein, all other values will be the same as those of the reference solution as summarized in table I. It is significant that the shape of the curves is similar. For target miss distances $|dP|$ of 0.005 to 0.010 radius, the error due to approximating with 5 instead of 10 intervals is about 4 percent in success probability. Larger differences occur at 0.002 radius, but they represent only a difference in $|dP|$ of 0.0005 radius (2 miles). The requirements

for control ΔV with the same cases are shown in figure 7(b), and again the shape of the curves is similar. The largest change in ΔV requirement for a given success is less than 10 percent, and with given ΔV the largest change in success is 6 percent. However, in the region of most interest, where high success is probable, the differences are smaller, say 2 percent in success probability with arbitrary control ΔV available.

The somewhat poorer approximation of error distributions using 5 intervals is accepted as expedient, in that computing time is reduced by half.

Shape of $\text{err}_{\dot{\theta}}$. - The rectangular error distribution of measurement $\dot{\theta}$ is to be compared to a standard-deviation distribution with equivalent mean error. Thus, half the errors are greater than the mean (± 0.0005 radian) in both cases. In addition, the dead-band size $\dot{\theta}_{\text{cal}}$ is 10^{-3} radian in both cases but has a different significance, since rectangular $\text{err}_{\dot{\theta}}$ allows no errors greater than $\dot{\theta}_{\text{cal}}$, while the "normal" $\text{err}_{\dot{\theta}}$ allows 9 percent of the readings to be greater than the dead band. The effect of dead-band size is treated later; at this point, the closest reasonable equivalence is used for comparison.

A comparison of results is shown in figure 8(a) for accuracy of guidance. Here again the shape of the curves is similar, but the large maximum deviations (carried to 3σ) of the normal distribution of errors reflect in larger maximum errors in the final perigee distribution. The maximum error is 0.04 rather than 0.02 radius with rectangular $\text{err}_{\dot{\theta}}$. (The difference shown is smaller in the region of 0.005 to 0.010 radius than would be shown if a 10-interval approximation were compared instead of 5 intervals.) The normal distribution is approximated by 11 intervals of equal argument. Figure 8(b) shows that about 50 percent more ΔV is required with a standard deviation of measurement errors to obtain the same success probability. The additional ΔV required is anticipated with normal distributions, since the large maximum deviations permit errors larger than $\dot{\theta}_{\text{cal}}$, and some ΔV is expended in oscillation of the vehicles' path around the target trajectory.

As a consequence of the substantial effects on guidance of normal relative to rectangular distributions of measurement errors, some of the guidance effects are evaluated with both assumptions.

Size of $\text{err}_{\dot{\theta}}$. - The magnitude of measurement errors is, at the same time, one of the most important and most uncertain factors in the guidance problem considered. This uncertainty is reflected in the range over which the assumed errors are varied, namely, from 0.0001 to 0.01 radian per unit dimensionless time. In order to maintain an equivalent guidance scheme

over the range of errors, the dead-band size $\dot{\Theta}_{cal}$ is varied with the maximum error in angular-rate measurement. The rectangular distribution is used, and the maximum error defines the size of the distribution.

Results are shown in summary form in figure 9, where the probability of hitting the target is a function of measurement-error size for constant values of $|dP|$. The obvious trend toward higher probabilities ("success") with smaller errors is indicated over the entire range from 10^{-4} to 10^{-2} -radian maximum error. Measurement errors of 10^{-4} radian permit guidance to within 0.0025 radius of the target 100 percent of the time (fig. 9(a)). For errors of the size used in the reference solution, 0.001 radian, all of the vehicles are guided to 0.02 radius or better, and half to within 0.0025 radius of the target perigee, where 0.02 is the altitude of the perigee above the surface of the planet. With even larger err^{Θ} , some vehicles of the family will follow trajectories below the surface of the planet; for instance, if errors of 10^{-2} are considered, less than 12 percent will be within the 0.02-radius miss distance, so that about half of the rest (40 percent) will crash.

The requirements for ΔV over the range of err^{Θ} are shown in figure 9(b). The probability of success is the percent of vehicles requiring less ΔV capability than the assumed availability. As err^{Θ} increases, the success with given availability decreases sharply: With small errors (10^{-4} radian), the approach is executed with high success using ΔV of the order 0.05 to 0.10 escape velocity; while with errors of 10^{-2} radian, more than $0.3 V_e$ is required.

With the logic treated in this analysis, it appears unlikely from either accuracy of guidance or ΔV requirement considerations that guidance with high success is practical with measurement errors much greater than 0.001 radian per unit dimensionless time. There are, naturally, many factors to consider in the logic of guidance and control, and most effects depend on the type of errors encountered. The errors used here are considered to represent a conservative estimate of accuracies anticipated for this type of interplanetary mission (ref. 6 or 7).

Initial perigee distribution, err^{P_i} . - Without prior consideration of launch and/or midcourse guidance, it is necessary to assume initial errors in the trajectories of the family of vehicles. The assumed initial distributions are broken into increments as described previously.

In addition to rectangular distributions with maximum errors of ± 5 , ± 10 , and ± 20 radii, a normal distribution with a mean error of ± 5 radii is investigated. The normal distribution is considered a variation in shape using the same mean error as the ± 10 -radii rectangular distribution (since the rectangular distributions are referred to by maximum rather

than mean error). The probable success in guidance accuracy with this range of initial trajectory errors is shown in figure 10(a). No effect is indicated.

Velocity-increment requirements for the same initial err^P are shown in figure 10(b). The ΔV required to correct the trajectories to the target at 100-radius range is indicated as the ideal requirement. Larger initial errors caused larger ΔV requirements, as anticipated. The significance of the initial error assumptions is in the difference between the ΔV requirement with guidance and the ideal ΔV requirement at constant success probability. This term is representative of the cost of measurement errors and is shown in figure 10(c) with constant values of success probability as a function of the maximum error in the initial perigee distributions. Within accuracy limits of ± 5 percent, the cost of guidance over the cost of ideal correction is constant for the errors investigated.

Since the ideal correction is readily computed for any given err^P_i , it is possible to estimate the total cost of guidance once the cost for any perigee distribution is known. This result is significant to this analysis because guidance considerations are then insensitive to particular assumptions. Consequently, a greater generality is maintained within the limits considered; namely, initial errors from ± 5 to ± 20 radii ($\pm 20,000$ to $\pm 80,000$ miles).

Trajectory Considerations

For the two-dimensional problem treated here, using (just the range of) the perigee as target, the desired trajectory is defined by only two parameters: the energy E , and the perigee P_{tar} .

It is in general difficult to compare guidance with different values of E or P_{tar} on an equivalent basis, since the relations of distance and time are functions of both parameters. For example, as energy changes, the time to the perigee also changes, and the use of a fixed sampling rate schedule results in sampling at different ranges and different numbers of times. The variation of a system parameter therefore leads directly to modified guidance logic. A comparison is nevertheless useful, because it is a prime objective to indicate the sensitivity of the assumptions necessary to formulate a guidance problem.

Initial energy. - A comparison of guidance accuracy for initial-energy assumptions of 0, 0.1, and 0.2 is shown in figure 11(a). An initial range of 100 radii is used in each case; thus, initial time to go and the number of sampling points are less for higher-energy trajectories, since the sequence of time increments is the same as in the reference solution.

The trend due to energy change alone was indicated in the analysis using maximum error coefficients (fig. 4): For $R \rightarrow P_{tar}$ the errors expected were similar over the range of E . The statistical solution verifies this expectation in that no consistent trend with variation of E is indicated.

The ΔV requirement is dependent on the errors over the range of R , not just those near the target that influenced the guidance accuracy. As energy increases, the errors decrease (fig. 4), so that some correction should be accomplished at longer range with hyperbolic trajectories. On the other hand, the correction of errors for higher-energy trajectories requires more ΔV . The result of this interaction is shown in figure 11(b) for the parabolic and hyperbolic cases. The success for a given ΔV is less with hyperbolic trajectories, but the trend is not consistent, in that the $E = 0.2$ curve is above the $E = 0.1$ curve. The undetermined effect of altered sampling rate should be counter to this trend, since less sampling points are used at $E = 0.2$.

It is not considered important in this analysis to establish the details of these effects. What is important, however, is that the assumption of trajectory energy is not leading to unique results, applicable only to the particular assumed value.

Target perigee. - Analysis of error coefficients indicated that the accuracy around the target perigee P_{tar} is dependent on the range of P_{tar} to the $5/2$ power (approximately) and that "good" guidance is accomplished only as the range R becomes small. The uncertainties due to altered time are very small when perigee is varied and, though present, are not important in affecting the sampling rate.

The probability of hitting a fixed target band is shown as a function of target perigee in figure 12(a). As expected, high success is attained with small target miss distances only if P_{tar} is close to 1 radius. For P_{tar} at 20 radii, errors of greater than 1 radius must be acceptable if high success is desired. It is probably more realistic to consider errors as proportional to the range of P_{tar} , since the high accuracies required near the surface would most likely not be needed farther away. From the analysis of maximum error coefficients, the parameter dP/P_{tar} should vary approximately as $P_{tar}^{3/2}$. This trend is vaguely indicated in figure 12(b), where proportional miss distance dP/P is shown, again as a function of P_{tar} . However, for P_{tar} up to possibly 5 radii, the adverse effects are not unreasonable; for instance, 90-percent success is possible with $|dP/P|$ of 2 percent. Thus, for an Earth approach at a range of 20,000 miles, vehicles would be within ± 400 miles 90 percent of the time.

The ΔV requirements over the range of P_{tar} from 1 to 20 radii are shown in figure 12(c). Since the ΔV required to correct a given error in trajectory is roughly proportional to $1/R$ (ref. 16), and since the dead band is larger in effect so that fewer corrections are made with large P_{tar} , the ΔV required should be less. This is indicated by the decreasing success probability at constant ΔV available as P_{tar} decreases.

Without additional assumptions or knowledge, it is difficult to assess the significance of these effects. Many mission objectives will not permit significant free choice of P_{tar} ; for instance, atmospheric drag or soft landing approaches. On the other hand, the choice of a "turning" pass around a planet for the purpose of altering a heliocentric orbit would be sensitive to perigee errors, which would be large if the turning angle were small. In any case, guidance considerations are sensitive to the target perigee used in any particular mission. Herein, emphasis is placed on those missions desiring a low-altitude approach. As a consequence, a loss in generality is incurred by assuming P_{tar} equal to 1.02 radii.

Control Functions

At this point it is of interest to consider the control functions needed to execute guidance for the problem as hypothesized in preceding sections. The errors and trajectory parameters used are summarized in table I, and the significance of the particular values has been considered.

Dead-band size, $\dot{\theta}_{cal}$. - The need for efficiency in controlling the expenditure of ΔV and the logic leading to the definition of dead-band size in terms of measurement-error calibrations were illustrated in the analysis. Since the logic of guidance and control depends on the relation between the $\dot{\theta}_{cal}$ and the $err\dot{\theta}$, there is doubt whether rectangular distributions can adequately represent a realistic situation. Therefore, the results will be shown for solutions using both rectangular and standard-deviation error assumptions.

The summary form of results is used in figure 13(a) to show the effect of $\dot{\theta}_{cal}$ on guidance accuracy and ΔV requirements using rectangular measurement errors. With the $\dot{\theta}_{cal}$ equal to 10^{-3} radian, precisely 100 percent of the errors in measurement fall within the dead band. With $\dot{\theta}_{cal}$ smaller than 10^{-3} radian, less "damping" is expected, since some readings fall outside the dead band and are treated by guidance logic as trajectory errors. On the other hand, with $\dot{\theta}_{cal}$ greater than 10^{-3}

radian, the system is essentially "overdamped," since trajectory errors will be overlooked in logic and considered measurement uncertainties. Thus, for larger $\dot{\theta}_{cal}$ the probability of success in hitting a target of given size is reduced, since less of the "indicated" error is acted on. If a target miss distance of ± 0.010 radius (± 40 miles) is desired, the success probability with $\dot{\theta}_{cal}$ equal to 1.8×10^{-3} radian is only 50 percent.

As might be anticipated, the probability of success in performing the approach with given ΔV available is generally highest if the dead band is exactly the expectation of measurement errors. Thus, too little dead band allows oscillation, and too much delays correction to smaller range where execution is more expensive. It is important to note that success probability is rapidly reduced by changing $\dot{\theta}_{cal}$ either direction from the optimum; for example, with $0.15 V_e$ available, reducing $\dot{\theta}_{cal}$ by 0.20 reduces success probability by more than 20 percent.

The same form of results is shown in figure 13(b) using standard-deviation measurement-error assumptions. Allowing for the expected numerical differences (fig. 7(b)) between results with different $err_{\dot{\theta}}$ shape, the trends with $\dot{\theta}_{cal}$ variation are remarkably similar, even to the point at which the optimum success probability with given ΔV occurs. With the normal distribution of measurement errors the expected errors in measurement will fall within the dead band 90.8 percent of the time for $\dot{\theta}_{cal}$ of 10^{-3} radian. The optimum $\dot{\theta}_{cal}$ in terms of ΔV requirements is thus expected to be twice the mean error, or 1.35σ .

Three conclusions affecting guidance logic are thus indicated. First, the optimum values of $\dot{\theta}_{cal}$ due to accuracy and ΔV requirement differ so that arbitration between effects is necessary. It is also significant that the shape used in measurement-error distributions has little effect on the choice of optimum values of $\dot{\theta}_{cal}$.

Thirdly, and possibly most important, the $\dot{\theta}_{cal}$ desired is to be based on instrument calibration data, so that to obtain the desired guidance it is mandatory to get accurate calibration. For instance, if conservative estimates of errors are used, the effect will be to use higher than expected $\dot{\theta}_{cal}$, thus moving to larger target miss distances and larger ΔV requirements; in this way any advantage gained because the errors are smaller than the "estimate" is more than compensated by the mismatch of guidance logic caused.

Restraint factor K. - It was indicated in the analysis that the control logic should include factors to determine what part of errors not uniquely distinguishable as trajectory errors should be corrected by ΔV

application. The fraction to be acted upon is described by the restraint factor K , such that $K = 0$ logic would correct any indicated error, while $K = 1$ would correct only the trajectory error distinguishable from measurement errors by use of $\hat{\Theta}_{cal}$.

The effects of variation of K from 0 to 1 are shown in figure 14. The success of hitting a target of given size decreases as restraint increases, while success with given ΔV availability generally increases. However, for restraint factors above about 0.8 the success decreases if 0.1 V_e is available. The large scatter is due to a major extent to the complex combination and interaction between nonlinear effects.

Let it suffice at this point to indicate that significant conflicting effects due to variation of restraint factor are expected. Again, an analysis of greater scope is required to arbitrate the costs of target miss and ΔV expenditure and so to define in detail the optimum values of K .

Sampling rate, $\Delta\tau_{go}$. - In interpreting the significance of sampling-rate variations it is important to remember that smoothed data are assumed. As a result, the constant distributions may actually represent modified instrument capability.

The results of variation in sampling rate are shown in figure 15 as a function of the percentage decrease in time to go between sampling points. The minimum increment is 3.0 units for the reference solution using 20 percent $\Delta\tau_{go}$, and the relation is maintained essentially constant by using minimums of $2\frac{1}{4}$ for 15 percent, $1\frac{1}{2}$ for 10 percent, and $3/4$ for 5 percent $\Delta\tau_{go}$. The resulting number of corrective actions could then be as many as 23 (reference), 37, 47, and 95 (conditional optimum) for $\Delta\tau_{go} = 20, 15, 10$, and 5 percent, respectively.

As the time between data sampling decreases and the number of sampling points increases, the success with given target size and given ΔV available increases. For example, for a $|dP|$ of 0.0025 radius (10 miles), the success increases from 45 to 80 percent as $\Delta\tau_{go}$ is reduced from 20 to 5 percent. Also, for an available ΔV of 0.1 V_e , the same change in sampling rate increases success by 25 percent. The reference solution is anything but an optimum so far as sampling rate is concerned. This deviation is expedient from calculation-time considerations. Since the minimum increment of $3/4$ unit on approach to Earth would represent about 7 minutes, still shorter intervals could be considered.

Caution should be exercised in applying these results to guidance using smaller dead band or less restraint than are used here ($\hat{\Theta}_{cal} = 10^{-3}$

with error maximum of 10^{-3} and $K = 1.0$), since they are definitely not expected to have the same effects.

Initiation of guidance $\tau_{go,i}$. - The most important effect to consider in establishing the range of initiation of guidance is the radar requirement in terms of power and equipment capabilities. This is beyond the scope of investigation here, but it is readily determined what costs in terms of ΔV and accuracy are assessed when range is reduced.

Results of guidance are shown in figure 16 as a function of the range of initial sampling. It is immediately obvious that guidance accuracy is unaffected by reducing initial range from 100 to 20 radii. However, the probability of success for given ΔV availability is reduced by reduction in R_i ; for example, with 0.15 escape velocity of ΔV available, 95-percent success is obtained when guidance is initiated at 100 radii (400,000 miles), while 89-percent success results with initiation at 40 radii and only 60-percent success is obtained starting at 20 radii (80,000 miles).

Cutoff time, $\tau_{go,min}$. - It was indicated in the analysis that errors in perigee varied approximately as $R^{5/2}$ (eq. (15)). This trend was verified in a gross sense by all statistical evaluations and illustrates the necessity of using active control all the way to the target (i.e., no cutoff).

Limitations on velocity-increment size. - The remaining control function is to maintain the application of individual ΔV within the limits specified. Fuel losses associated with very small ΔV from large engines may be one cause for limiting the minimum ΔV allowable, while cooling problems may limit the maximum ΔV supplied by any one impulse. It is of interest here to assess the cost in guidance performance attributable to limited range of ΔV impulses available.

Consider first the limitation of minimum ΔV . Control logic acts to omit the use of ΔV when impulses less than the minimum are demanded. Results of guidance are shown in figure 17 as a function of ΔV_{min} over the range from 0 to $0.05 V_e$. For limitations up to $0.001 V_e$ such as assumed in the reference solution, no significant adverse effects are indicated. However, above $0.001 V_e$ there is a gradual decrease in success in terms of both accuracy and ΔV requirements. The cost in success of limiting ΔV_{min} to $0.05 V_e$ is about 45 percent if $|dP|$ of 0.01 radius is allowable, while the same limitation would reduce success with $0.1 V_e$ about 40 percent.

In rough terms then, limiting the minimum ΔV to 0.01 escape velocity would cost about 15 percent in both accuracy and thrust availability success, while a limit of 0.001 V_e would have no significant effects.

If, on the other hand, maximum limits are placed on ΔV impulses, the guidance results are affected as shown in figure 18. A limit of 0.1 V_e is essentially unlimited, since guidance does not demand ΔV larger than 0.1. Reducing $\Delta V_{\max}$ to 0.05 V_e has little effect on either accuracy or total ΔV expenditure success, but further reductions result in rapid deterioration of guidance accuracy success, and in ΔV success if low ΔV is available.

Using $\Delta V_{\max}$ of 0.05 and $\Delta V_{\min}$ of 0.001 escape velocity leaves a demand for impulse size of 50/1 without cost in performance. Further reduction in ΔV range is available at some expense in guidance and control performance.

Finally, it is conceivable to consider the use of a small number of fixed-impulse engines that would be used as a "quick" method of control. From the previous discussion, where a range of ΔV of 50 to 1 was shown to be desirable, it seems unlikely that one fixed size would be satisfactory. However, results are shown in figure 19 for impulses of 0.01, 0.025, and 0.05 V_e . From accuracy considerations no significant success is possible without very large target sizes; for example, ± 0.2 radius (800 miles). With these large target sizes, large ΔV capability is needed to permit reasonable success.

In a gross sense, again, the cost of using fixed-impulse engines is of the order of a decade. Though it is readily argued that more refined logic would improve guidance, any improvements might just as easily be argued for the variable-impulse guidance. It must be concluded that large penalties will be paid if fixed-impulse engines are used.

CONCLUSIONS

Guidance and control of the approach phase appear feasible if errors in the gyroscopic measurements of trajectory angular rate are of the order 10^{-3} radian or smaller. This would represent drift rates of about 20-second arc per minute for an approach to Earth, and is of the order of current expectations. With this measurement accuracy, vehicles can be guided to ± 0.004 radius (16 miles) at the target with 98.5-percent success if supplied with 0.1-escape-velocity control capability, or 99.9 percent with 0.14 V_e (3500 or 5000 ft/sec). This accuracy at the target is indicated by reference 16 to be of the order required to permit atmospheric-drag decelerations at Earth or Venus with 20-g maximum deceleration and no lift or with 5-g maximum deceleration using modulated lift with a lift-drag ratio up to 1.0.

Residual errors from the midcourse control become initial errors in trajectory for the approach phase and have little effect on guidance accuracy within the range ± 20 radii, or $\pm 80,000$ miles, at least. Initial errors have a predictable effect on ΔV requirements; namely, the ΔV cost of instrument errors is roughly constant over the range of perigee errors. The energy level of the mission was found to be of minor significance to control over the range from parabolic to $E = 0.2$ because of compensating effects in measurement sensitivity and sampling rate. On Earth, this would compare to surface velocities of about 35,000 to 39,000 feet per second. The target, defined in terms of an altitude (range) and a direction of rotation but with undefined orientation, should be less than about 1 radius above the surface if accuracy is desired. Higher targets lead to larger miss distance, even in terms of errors as a percentage of altitude (range). However, at high altitude less control thrust is required.

Guidance logic including a dead band and a restraint on thrust application is useful if not necessary in reducing total thrust expenditure, but has the disadvantage that guidance accuracy is reduced. The specific size of dead band and degree of restraint again would depend on arbitration of effects beyond the scope of this analysis. It is noted that the use of measurement instrument calibrations in defining dead-band size leads directly to a need for precise calibration data; and, in general, poor "estimates" of instrument capability would result in seriously reduced mission-phase success. To the limits of digital accuracy and computing time encountered in solution, the guidance and control were significantly improved by increasing the sampling rate. However, the maximum number of data acquisition points considered was less than 100, and the minimum increment between corrections was $3/4$ of a unit (7 min).

The thrust devices used in control were found to require a variation in impulse size from 0.001 to 0.05 escape velocity, with few large impulses (up to 0.05 or 1750 ft/sec) and large numbers of small impulses (about 0.001 or 35 ft/sec). Penalties in guidance accuracy or total ΔV expenditure would result from attempts to reduce the range of variation.

Lewis Research Center

National Aeronautics and Space Administration
Cleveland, Ohio, December 7, 1959

APPENDIX A

SAMPLE CALCULATIONS

The use of arbitrary time increments in correction logic requires solution for the range R in terms of time. To this end,

$$H = R^2 \dot{\phi} = R^2 \frac{d\phi}{dt}$$

$$R = \frac{2H^2}{1 + \epsilon \cos \phi}$$

and, parallel to references 17 and 18 and many others,

$$\int_0^{\tau_{go}} dt = \frac{1}{H} \int_0^{\phi} R^2 d\phi$$

where limits are chosen to yield the time from $R = R_1$ to the perigee P . Integration yields after simplification

$$\tau_{go} = \frac{2}{3} \sqrt{R - P} (R + 2P) \quad E = 0$$

$$\tau_{go} = \frac{H \tan \gamma}{E} - \frac{1}{2E\sqrt{-E}} \cos^{-1} \left(\frac{1 + 2ER}{\epsilon} \right) \quad E < 0$$

$$\tau_{go} = \frac{H \tan \gamma}{E} - \frac{1}{2E\sqrt{E}} \ln \left(\frac{1 + 2ER + 2\sqrt{E} H \tan \gamma}{\epsilon} \right) \quad E > 0$$

where

$$H \tan \gamma = \sqrt{R - P} \sqrt{(R + P)E + 1}$$

These relations must be solved iteratively for R knowing E , τ_{go} , and P . (Effort is required to obtain convergence over the likely range of variables and trial values.)

Thus, for any vehicle with energy E_a , at time $\tau_{go,a}$, and on a trajectory leading to perigee P_a , range R_a is obtained. Then

$$\begin{aligned}
 V_a &= \sqrt{\frac{1}{R_a} + E_a} && \text{from energy (1)} \\
 \left. \begin{aligned}
 H_{tar} &= + \sqrt{P_{tar}(P_{tar}E_{tar} + 1)} \\
 H_a &= \pm \sqrt{P_a(P_aE_a + 1)}
 \end{aligned} \right\} && \text{from (3)} \\
 \left. \begin{aligned}
 \dot{\theta}_{tar} &= H_{tar}/R^2 \\
 \dot{\theta}_a &= H_a/R^2
 \end{aligned} \right\} && \text{from (2) noting that } R = R_a
 \end{aligned}$$

taking advantage of the fact that $E_a \approx E_{tar} \approx E$ and basing logic on the target trajectory.

Signs of $\dot{\theta}$ and H are determined by the direction of rotation about the planet. (Artificial signs are used for this purpose and are associated with dP for convenience.) Also,

$$\gamma_a = \cos^{-1} \frac{H_a}{RV_a} \quad \text{from (2), where } 0 \leq \gamma \leq \pi \text{ for inward flight}$$

$$\dot{R}_a = V_a \sin \gamma_a \quad \text{from components of } V_a$$

With these known and desired conditions, the "imagined" trajectory is fully defined by

$$\dot{\theta}_i = \dot{\theta}_a + \text{err} \dot{\theta}_n$$

where $\text{err} \dot{\theta}_n$ represents a possible error in the assumed measurement spectrum.

For a guidance scheme incorporating a dead band and restraint, corrective thrust velocity is to be used if

$$|\dot{\theta}_i - \dot{\theta}_{tar}| \geq \dot{\theta}_{cal}$$

and

$$\dot{\theta}_2 = \pm K \dot{\theta}_{cal} + \dot{\theta}_{tar}$$

where $\dot{\theta}_{cal}$ represents the calibrated or designed error expectation, and the restraint factor K with the expected error $\dot{\theta}_{cal}$ describes the uncorrected portion of error.

If thrust is tentatively to be used,

$$r_i = \tan^{-1} \frac{\dot{R}_a}{R\dot{\theta}_i} = \tan^{-1} \left(\frac{V_a \sin r_a}{R\dot{\theta}_i} \right)$$

$$V_i = \frac{\dot{R}_a}{\sin r_i}$$

$$r_{2,i} = \cos^{-1} \left[\frac{R}{V_i} (\dot{\theta}_2) \right]$$

Thrust-velocity vector magnitude is then

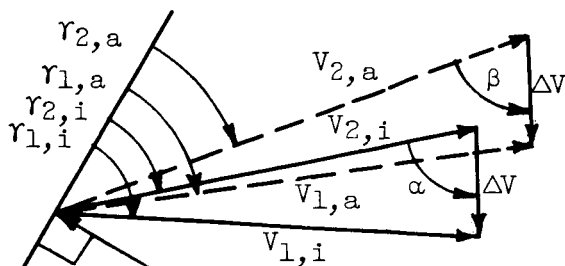
$$\Delta V = 2V_i \sin \frac{(r_2 - r_1)_i}{2}$$

and is tested prior to application for impulse size limitations imposed by control logic:

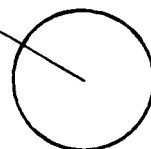
If $\Delta V > \Delta V_{\max}$, set $\Delta V = \Delta V_{\max}$

If $\Delta V < \Delta V_{\min}$, set $\Delta V = 0$

Since both the magnitude and direction of ΔV result from treatment of imperfect data, the effect of the thrust application is found from the following:



$$R_{1,i} = R_{2,i} = R_{1,a} = R_{2,a} = R$$



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$$\beta = \frac{\pi}{2} - \frac{r_{2,i} - r_i}{2} + (r_i - r_a)$$

$$V_2^2 = \Delta V^2 + V_a^2 - 2V_a \Delta V \cos \beta \quad [\text{Law of cosines}]$$

$$E_2 = V_2^2 - \frac{1}{R} \quad [\text{Energy eq. (1)}]$$

$$r_2 = \sin^{-1} \frac{\Delta V}{V_a} \sin \beta \quad [\text{Law of sines}]$$

$$H_2 = V_2 R \cos r_2 \quad [\text{Conservation of momentum (2)}]$$

$$\left. \begin{aligned} P_2 &= H_2^2 && \text{if } E_2 \rightarrow 0 \\ P_2 &= \frac{1}{2E_2} \left(\pm \sqrt{1 + 4E_2 H_2^2} - 1 \right) && E_2 \neq 0 \end{aligned} \right\} \quad [\text{Eqs. (4)}]$$

where solution of P_2 requires significant figures in E (i.e., $|E| > 10^{-5}$) for choice of solution, since $E = 0$ to eight figures is not likely for digital reasons.

The definitive parameter P_2 represents the trajectory of the vehicle after thrust application. By assumption of like energy vehicles, as discussed, P_2 serves to group vehicles $P_n < P_2 < P_{n+1}$ in tabular form. The frequency of occurrence is accumulated $f_{n,n+1} + f_2 \rightarrow f_{n,n+1}$ by continuous summation.

APPENDIX B

ERROR ANALYSIS OF RANGE - RANGE-RATE SCHEME OF MEASUREMENT

A scheme of navigation using radar or radio instrumentation with measurements of range and range rate requires at least two successive points along the trajectory. Using the minimum information and eliminating angular rate $\dot{\theta}$ from the energy equation (1) by the momentum equation (2),

$$\left. \begin{aligned} E &= \dot{R}_1^2 + \frac{H^2}{R_1^2} - \frac{1}{R_1} \\ E &= \dot{R}_2^2 + \frac{H^2}{R_2^2} - \frac{1}{R_2} \end{aligned} \right\} \quad (B1)$$

Eliminating E and solving for H^2 give

$$H^2 = \frac{R_1^2 R_2^2}{R_1^2 - R_2^2} (\dot{R}_1^2 - \dot{R}_2^2) + \frac{R_1 R_2}{R_1 + R_2} \quad (B2)$$

but note that the sign of $H = \pm \sqrt{H^2}$ is not available from this information alone. Assume that some other source of information is used to determine the sign of H (i.e., the direction of rotation).

The error in perigee due to measurement errors is found by substitution into (3):

$$dP = \frac{2H dH - P^2 dE}{2PE + 1}$$

From equation (B1),

$$P^2 dE = 2P^2 \dot{R}_1 d\dot{R}_1 + P^2 \frac{2H dH}{R_1^2} - \frac{2H^2 P^2}{R_1^3} dR_1 + \frac{P^2 dR_1}{R_1^2}$$

Therefore,

$$dP = \frac{\left(1 - \frac{P^2}{R^2}\right) 2H dH - P^2 \left(2\dot{R}_1 d\dot{R}_1 - \frac{2H^2}{R_1^2} \frac{dR_1}{R_1} + \frac{dR_1}{R_1^2}\right)}{2PE + 1}$$

$$2H dH = 2H \frac{\partial H}{\partial R_1} dR_1 + 2H \frac{\partial H}{\partial \dot{R}_1} d\dot{R}_1 + 2H \frac{\partial H}{\partial R_2} dR_2 + 2H \frac{\partial H}{\partial \dot{R}_2} d\dot{R}_2$$

The partial coefficients are determined from (B2) as

$$2H \frac{\partial H}{\partial \dot{R}_1} = \frac{R_1^2 R_2^2}{R_1^2 - R_2^2} 2\dot{R}_1$$

$$2H \frac{\partial H}{\partial \dot{R}_2} = -\frac{R_1^2 R_2^2}{R_1^2 - R_2^2} 2\dot{R}_2$$

$$2H \frac{\partial H}{\partial R_1} = \frac{\dot{R}_1^2 - \dot{R}_2^2}{R_1^2 - R_2^2} \left[(R_1^2 - R_2^2)(2R_1 R_2^2) - R_1^2 R_2^2 (2R_1) \right] + \frac{(R_1 + R_2)R_2 - R_1 R_2}{(R_1 + R_2)^2}$$

$$\partial H \frac{\partial H}{\partial R_2} = \frac{\dot{R}_1^2 + \dot{R}_2^2}{R_1^2 - R_2^2} \left[(R_1^2 - R_2^2)(2R_1^2 R_2) + R_1^2 R_2^2 (2R_2) \right] + \frac{(R_1 + R_2)R_1 - R_1 R_2}{(R_1 + R_2)^2}$$

Substituting and simplifying,

$$\begin{aligned} dP = \frac{1}{2PE + 1} & \left\{ 2 \left(\frac{R_1^2 R_2^2}{R_1^2 - R_2^2} - P^2 \right) (\dot{R}_1 d\dot{R}_1 - \dot{R}_2 d\dot{R}_2) \right. \\ & + \left[\frac{R_2^2}{(R_1 + R_2)^2} - \frac{2}{R_2} \left(R_1 + R_2 - \frac{1}{R_1 - R_2} \right) \left(\frac{R_1^2}{P^2} - 1 \right) + 1 \right] \frac{P^2}{R_1^2} dR_1 \\ & + \left[\frac{R_1^2}{(R_1 + R_2)^2} - \frac{2}{R_1} \left(R_1 + R_2 - \frac{1}{R_1 - R_2} \right) \left(\frac{R_2^2}{P^2} - 1 \right) + 1 \right] \frac{P^2}{R_2^2} dR_2 \\ & \left. + \frac{2H^2}{R_2} (R_1^2 - P^2) dR_2 - \frac{2H^2}{R_1} (R_2^2 - P^2) dR_1 \right\} \quad (B3) \end{aligned}$$

The terms of equation (B3) vary considerably in relative magnitude, so that simplifying assumptions lead to poor approximations. The relation is solved digitally for the trajectory $P = 1$, $E = 0$ as used for figures 2 and 3. Results are shown in figure 20 using $R = R_1$ and $\Delta R = R_2 - R_1$; $dR_1 = dR_2 = d\dot{R}_1 = d\dot{R}_2$ is implied, but is not unreasonable for the comparison of trends desired here.

The minimum error coefficient for a given R occurs with ΔR of the same order as R . Thus (roughly), $R_2 = 2R_1$. This large an increase is inconsistent with the multiple correction scheme of interest here. For ΔR less than that of the minimum error, increases occur because of the sensitivity of differences $R_1 - R_2$ (or $R_1^2 - R_2^2$); while for larger ΔR , the R_2 terms contribute increasing error.

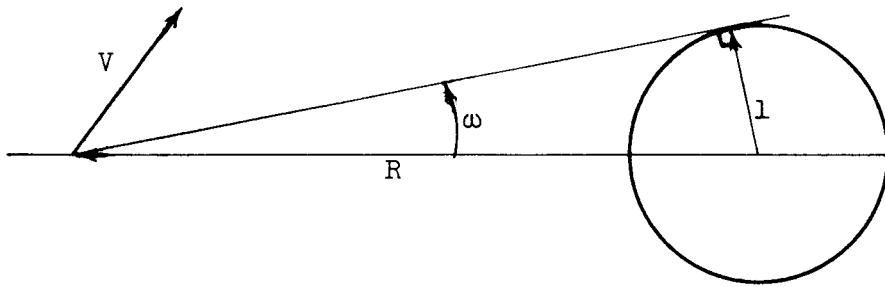
In order to evaluate the significance of $\Sigma |\partial P / \partial|$ over the range of R , impose now the ΔR sequence resulting from the sampling rate used in statistical solution (table I). Results are shown in figure 21 along with the sum of the absolute values of the curves of figure 2.

The significant conclusion is that the error in perigee determination is large at long range compared with the short-range error regardless of the type of instrumentation used. The variation of coefficient with range indicated in figure 21 agrees to a factor of 2 over the 4 decades of interest.

APPENDIX C

 APPLICABILITY TO OPTICAL OR INFRARED AND GYROSCOPIC
 SYSTEMS OF INSTRUMENTATION

Range R and range-rate $\dot{R}$ measurements can be determined by optical or infrared techniques using the apparent angle subtended by the target body. This problem is treated in reference 11, and only a simplified situation is considered here. With ω and $\dot{\omega}$ as independent measurements, where ω is the half-angle,



$$R = \frac{1}{\sin \omega}$$

$$\dot{R} = R \sqrt{R^2 - 1} \dot{\omega}$$

The error coefficients for perigee determination are found by

$$\frac{\partial P}{\partial \omega} = \frac{\partial P}{\partial R} \frac{\partial R}{\partial \omega} \quad \text{and} \quad \frac{\partial P}{\partial \dot{\omega}} = \frac{\partial P}{\partial \dot{R}} \frac{\partial \dot{R}}{\partial \dot{\omega}}$$

Thus,

$$\frac{\partial P}{\partial \omega} = \frac{1}{2PE + 1} \left[\left(2 - \frac{P^2}{R^2} \right) \frac{2H^2}{R} - \frac{P^2}{R^2} \right] R \sqrt{R^2 - 1} = \left[2 \left(2 - \frac{1}{R^2} \right) - \frac{1}{R} \right] \sqrt{R^2 - 1}$$

$$\frac{\partial P}{\partial \dot{\omega}} = \frac{1}{2PE + 1} (-2P^2 \dot{R}) (R \sqrt{R^2 - 1}) = -2(R - 1) \sqrt{R + 1}$$

$$\frac{\partial P}{\partial \dot{\omega}} = \frac{1}{2PE + 1} [(R^2 - P^2) 2H] = 2(R^2 - 1) \quad \text{as previously}$$

where the second form is reduced for the case $E = 0$, $P = 1$, and is shown in figure 22.

For numerically equal measurement errors, the angular-rate error does not dominate except at a range of close to 100 radii.

Assume now a finite difference determination of $\dot{\omega}$ and use the increment as the smallest used in the statistical analysis, 0.75 unit (or about $7\frac{1}{2}$ min on Earth approaches).

The relative instrument capability in angular measurements that would permit valid analysis assuming a single source of error may be roughly estimated, arbitrarily starting with $d\dot{\theta}$ as 10^{-3} radian per unit time (20-sec arc/min as a drift or about $2\frac{1}{2}$ -min arc each in different readings). To cause this error to dominate by a factor of 5 or more, $d\omega$ should be less than 10^{-4} radian (20-sec arc). The 10^{-4} -radian error with 0.75 unit between readings leads to $d\omega < 10^{-4}$ radian per unit time; so that, if the R accuracy is attained, then the $\dot{R}$ is also attained to sufficient accuracy.

It is concluded, therefore, that measurement systems based on 10^{-3} drift rate and 10^{-4} or less angular measurement error can be considered single error systems in effect. For this analysis let it suffice to indicate that some instrumentation schemes using optical or infrared instrumentation are accurately treated.

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APPENDIX D

STATISTICAL METHODS

Methods

The statistical operations used here are discussed in most texts on statistical mathematics (e.g., refs. 19 and 20). It is customary to use frequency polygons, or "distributions" of statistical data, with the area under the curve equal to 1 unit. For mechanical reasons the arguments used in this analysis are left in real numbers, though, as usual, the total frequency is assigned the value 1, or 100 percent. Therefore,

$$f_{\text{tot}} = \sum_1^n f_n = 1$$

so that the total frequency of the distribution is the sum of the frequencies of the increments f_n . Also,

$$\text{Area} = \sum_1^n (\Delta x_n f_n) \neq 1$$

where x can be any of the distributive arguments.

The simultaneous occurrence f_{mn} , where f_m is an increment of a frequency polygon independent (uncorrelated) of the frequency polygon of which f_n is an increment, is the product of the two frequencies:

$$f_{mn} = f_m f_n$$

and it follows that

$$f_{\text{tot}} = \sum_1^m \sum_1^n (f_m f_n) = 1$$

An example of the use of compound frequencies is the determination of trajectory errors. If a vehicle with a trajectory represented by err^P_n is guided by data with $\text{err}^{\odot}_m$, a single increment of the family is likely to occur f_{mn} percent of the time. Then, as all of the combinations are considered, the resulting family is obtained in terms of a new err^P frequency polygon with $f_{\text{tot}} = 1$.

In the process of treating, or "acting on," the family of vehicles, increments of ΔV are computed (if any are used). These increments are summed into tables of like-valued impulses by adding the frequencies. A frequency polygon results, representing the probability of using ΔV at a specific sampling point.

The total ΔV expended by the family of vehicles at the sampling point is then the sum of the individual ΔV times the frequency of their occurrence:

$$\Delta V_{\text{at one point}} = \sum_1^{mn} (\Delta V_{nm} f_{mn})_{\text{at one point}}$$

and the total ΔV for the approach is simply the total of the values at the sampling points. Also, since the average is the total divided by the number of vehicles and the number of vehicles is one,

$$\Delta V_{\text{tot}} = \Delta V_{\text{av}}$$

either for one sampling point or the mission.

The probability of using ΔV increments of a given size ΔV_j is the sum of the frequencies of ΔV_j at each sampling point k during the approach:

$$f_{j,\text{tot}} = \sum_1^k f_j$$

where f_j at each point is the sum of the frequencies f_{mn} which used ΔV impulses of magnitude ΔV_j .

It is desirable also to obtain the probability of vehicles using total ΔV during the approach. The combination of the probabilities of using ΔV at each sampling point into an overall probability for the approach is dependent on the relations between the successive impulses. This relation is not directly available from the calculations because of the contraction of data. However, since the successive ΔV applications are dependent on complex conic relations, logic, and uncorrelated errors, it is reasonable to consider the impulses uncorrelated. With this assumption, the frequency polygon at each sampling point k is successively combined with the previous combined probability and a "running total" is available at all points k .

In adding uncorrelated frequency polygons, the frequency of an increment of the resulting polygon is the product of frequencies, while the argument of the new increment is the sum of the arguments. The increments of the two frequency polygons are cross-multiplied (all combinations) in this fashion, and a new distribution is constructed by summing the frequencies of like-valued arguments.

Overall Success of Approach Phase

As shown in previous discussion, statistical measures of the mission-phase success are available in terms of ΔV requirements and guidance accuracy probabilities.

It is not obvious, however, what criterion is applicable in formulating a method of combination of the two probabilities into an "overall" probability. Three common relations between the distributions are considered, two of which are limiting cases:

(1) The maximum combined success will occur if the "unsuccessful" vehicles are the same in both distributions; that is, if failure in one distribution preempts failure in the other. Some numerical examples of combined successes with sketches of the two-dimensional distribution of points assumed in combination are shown in figure 23. If 90 and 80 percent are used as p_a and p_b , the combined probability p_c would be 80 percent, or $p_c = \text{the smaller of } p_a \text{ or } p_b$.

(2) The minimum combined success will occur if the unsuccessful vehicles in the two distributions are different. Thus, failure due to ΔV excludes the possibility of failure due to accuracy, and vice versa, and the number of failures is the sum of the failures due to ΔV and accuracy. If p_a and p_b were again 90 and 80 percent, the combined failure probability would be the sum of the failures, or the success would be 70 percent, from

$$p_c = p_a + p_b - 1$$

(3) The most common relation is that of uncorrelated probabilities, where, as earlier, the combined probability is the product of the components. Using 90 and 80 percent for p_a and p_b , the combined success is 72 percent, from

$$p_c = p_a p_b$$

The uncertainty in determination of p_c is basically the difference between the maximum and minimum values by different assumptions of

correlation. The maximum, or preemptive, correlation is not a realistic evaluation of the data expected in this analysis. The uncorrelated or exclusive assumptions lead to similar p_c that seem reasonable and that lead to conservative values of p_c .

This difficulty in finding correlation criteria and the unusual treatment of logic near the target (fig. 6(a)) make the use of combined probability for the mission phase somewhat doubtful.

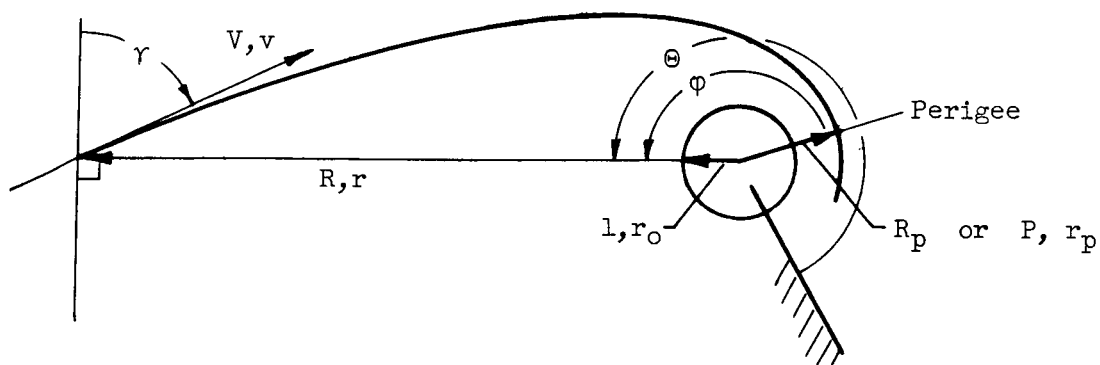
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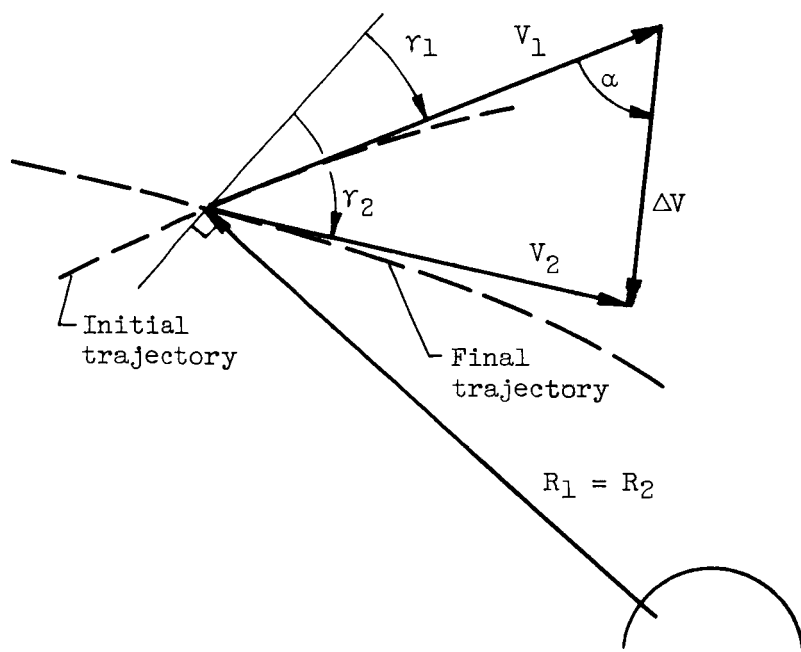
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20. Cramér, Harald: Mathematical Methods of Statistics. Princeton Univ. Press, 1946.

TABLE I. - ASSUMED PARAMETERS FOR CONDITIONAL OPTIMUM AND REFERENCE SOLUTIONS

Parameter	Assumed value		Approximate equivalent for Earth approach	
	Conditional optimum	Reference	Conditional optimum	Reference
Error distributions: Measurement error, $\text{err}\dot{\theta}$ Number increments, shape Size (max. error)	5, Rectangular 10^{-3} Radian		20-Sec arc/min drift or 2-min arc each of two readings with $\Delta t = 10$ min	
Initial perigee error, $\text{err}P_i$ Number increments, shape Size (max. error)	80, Rectangular 10 Radii		40,000 Miles	
Trajectory: Energy, E Target perigee, P_{tar}	0 (Parabolic) 1.02 Radii		----- 80-Mile altitude	
Control functions: Dead band, $\dot{\theta}_{cal}$ Restraint factor, K Initial time, $\tau_{go,i}$ Sampling rate, $\Delta\tau_{go}$: Use Percent τ_{go} or Min. increment Cutoff time, $\tau_{go,min}$ Maximum ΔV , ΔV_{max} Minimum ΔV , ΔV_{min}	10^{-3} Radian 1.0 676.8 ($R_i = 100$ Radii) 5.0 20.0 0.75 3.0 None, but sequence leads to last $\tau_{go} = 0.2$ Unlimited 0.001 Escape velocity		Same as $\text{err}\dot{\theta}$ ----- 4 Days (400,000 miles) 4-hr max. 20-hr max. 7-min min. 1/2-hr min. 2 Minutes ----- 35 ft/sec	



(a) Trajectory relations.



(b) Velocity relations.

Figure 1. - Notation used in analysis.

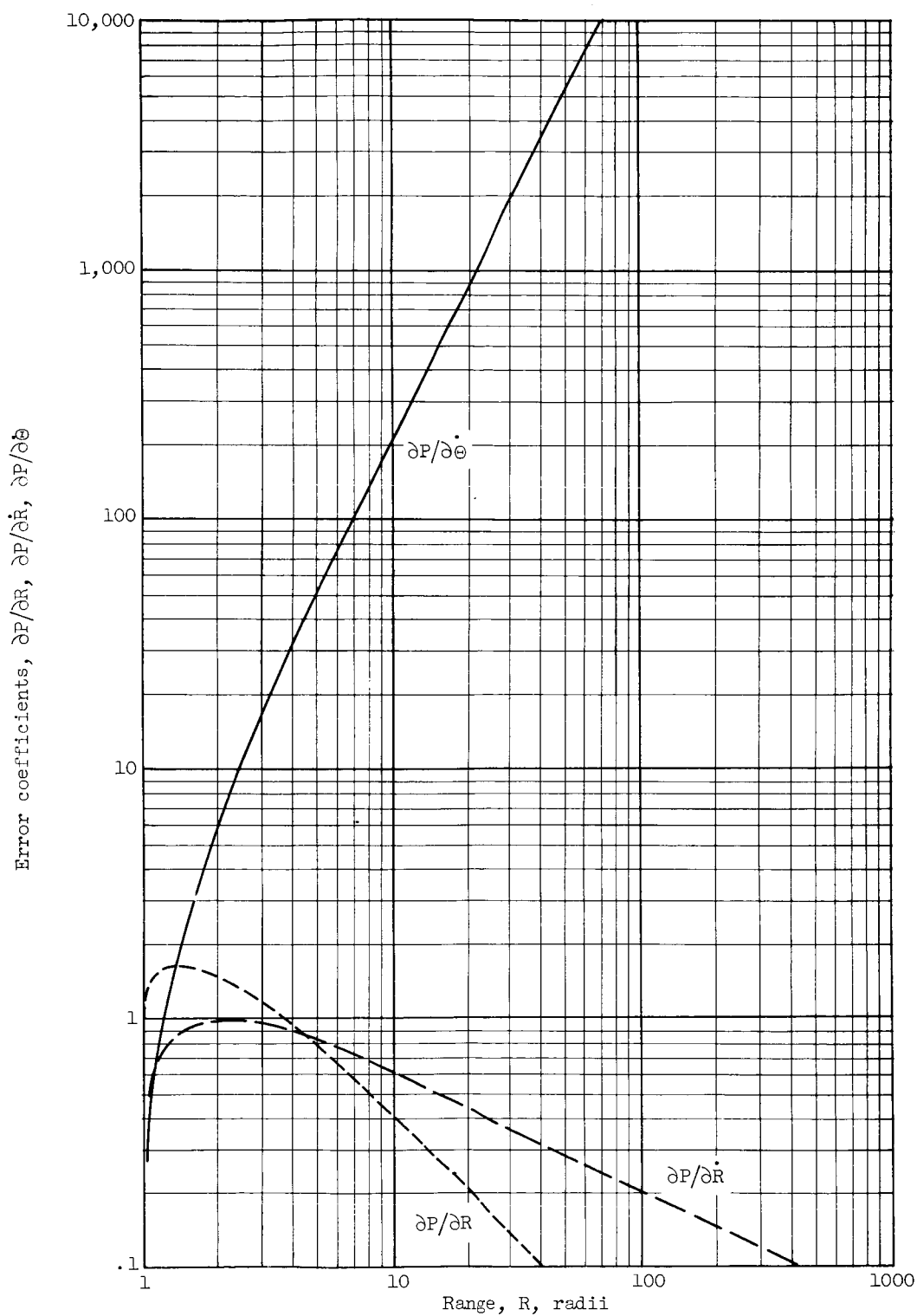


Figure 2. - Error coefficients of range, range-rate, and angular-rate measurements evaluated for energy $E = 0$ and perigee range $P = 1$.

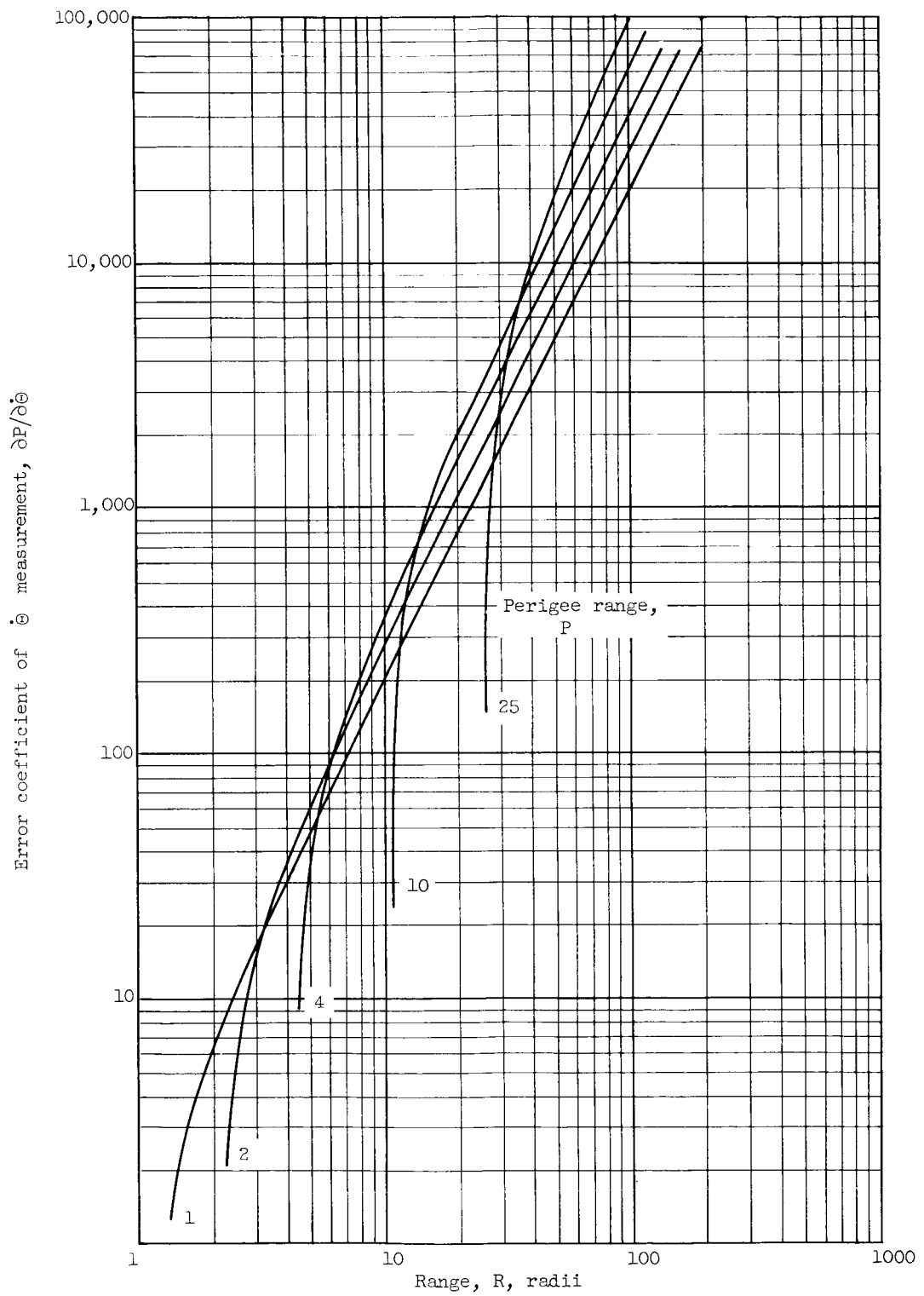


Figure 3. - Error coefficient $\frac{\partial P}{\partial \dot{\theta}}$ evaluated for energy $E = 0$ and a range of perigee values.

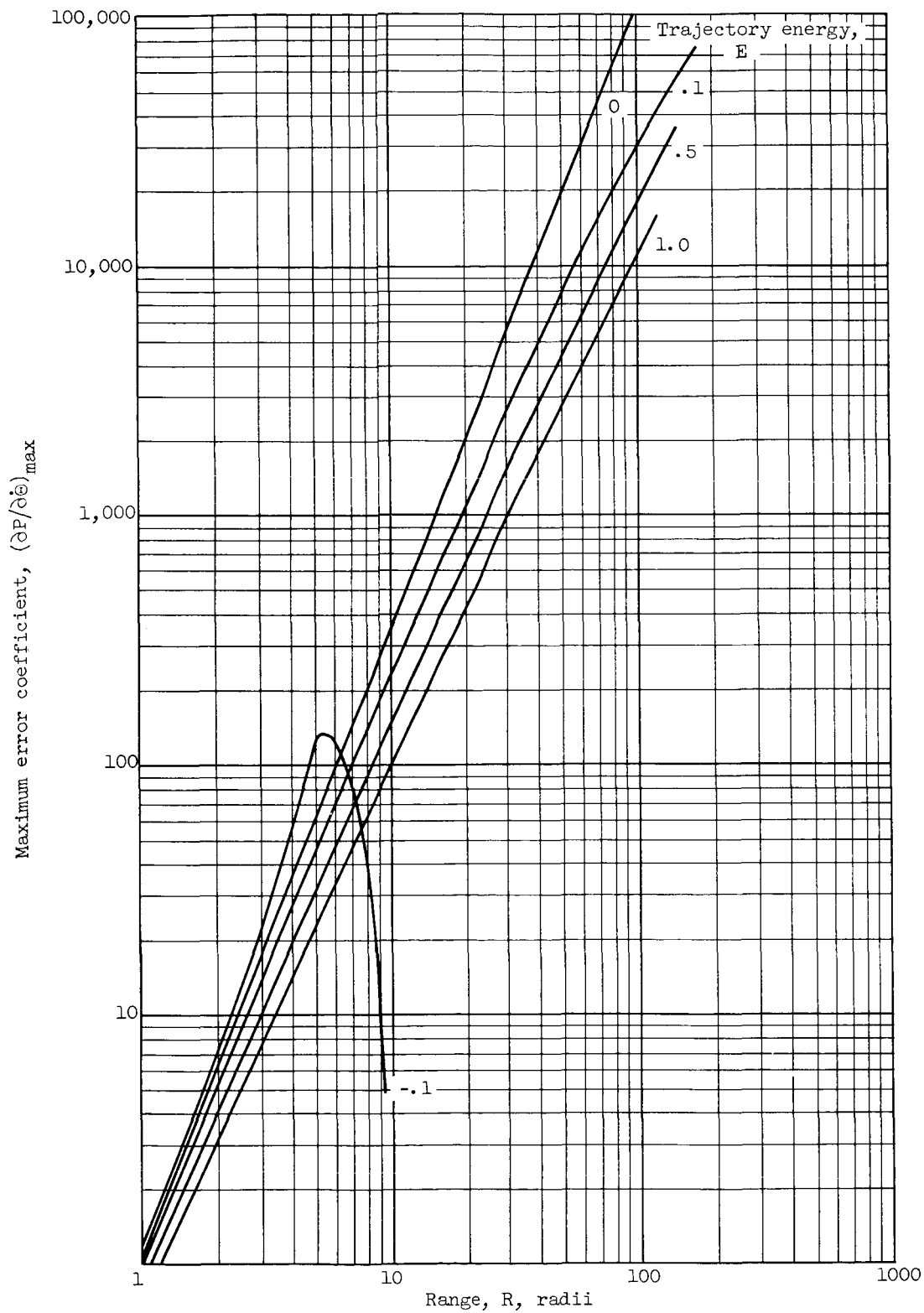
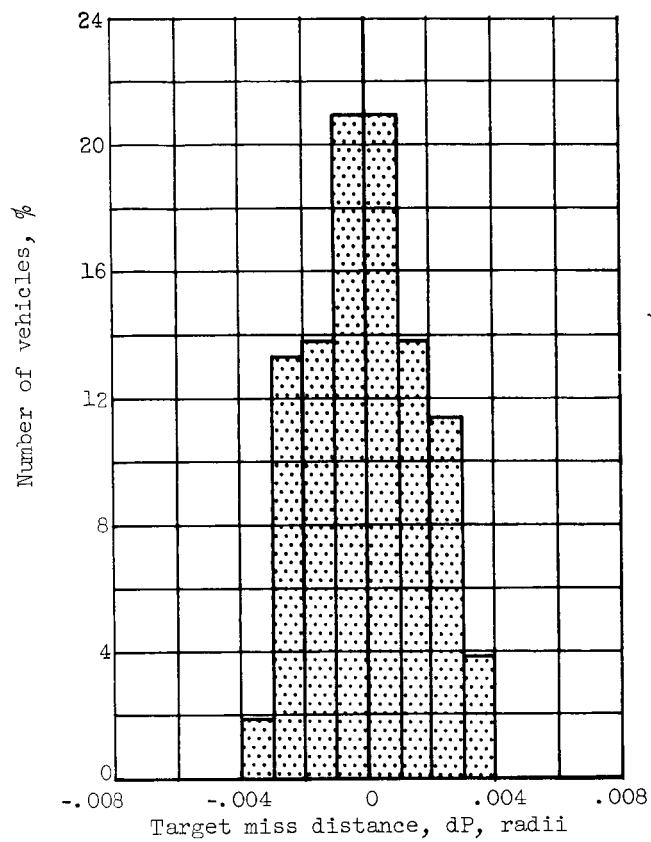
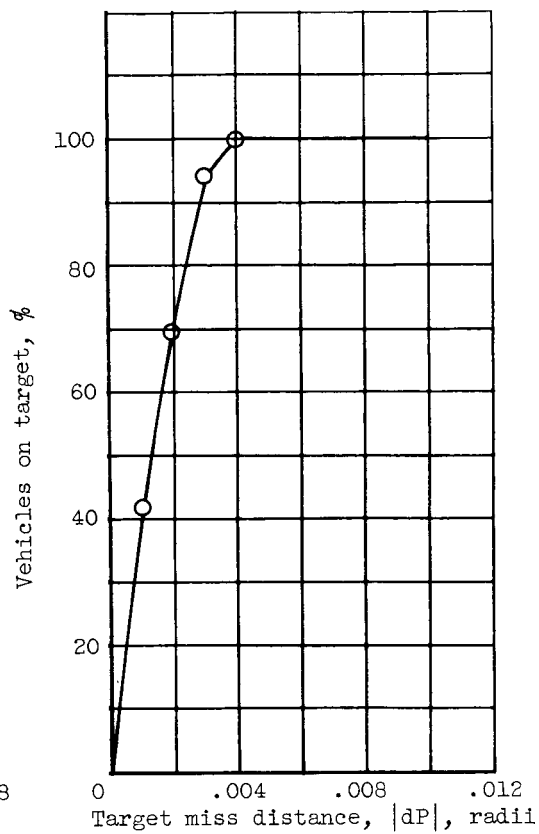


Figure 4. - Maximum error coefficient, $(\partial P / \partial \dot{\theta})_{\max}$.

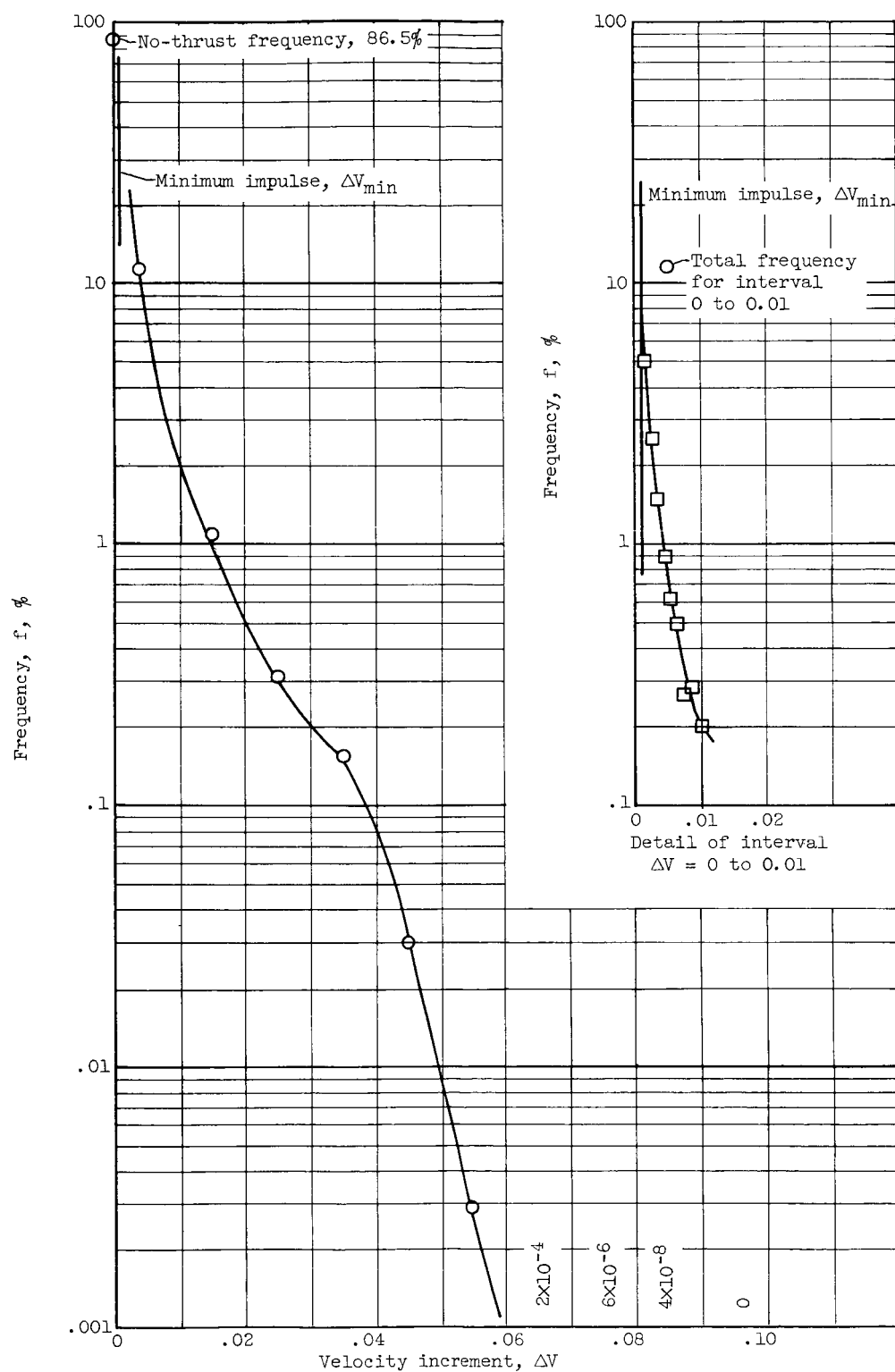


(a) Distribution of vehicles around target perigee after guidance.



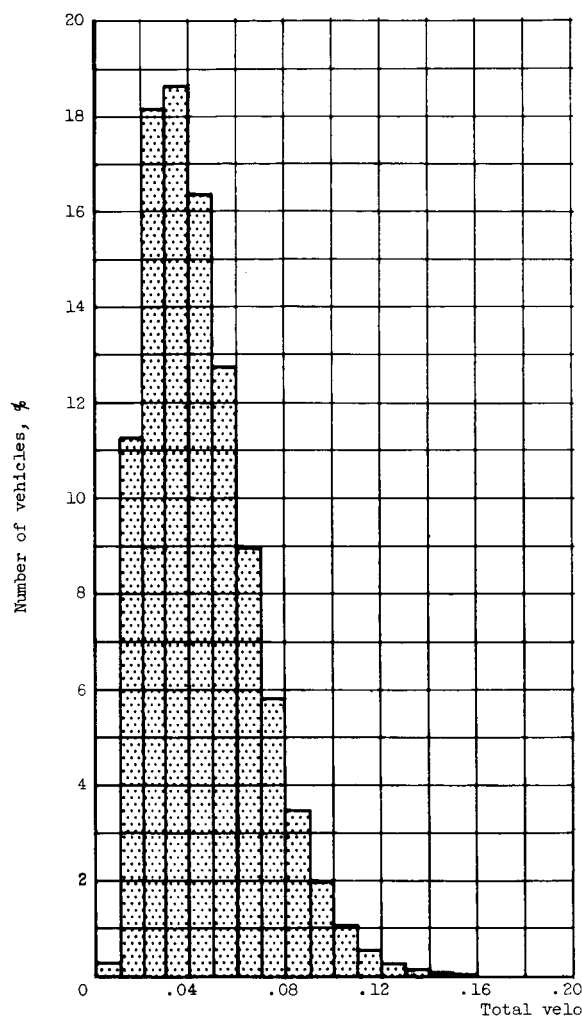
(b) Guidance accuracy.

Figure 5. - Results of conditional optimum solution.

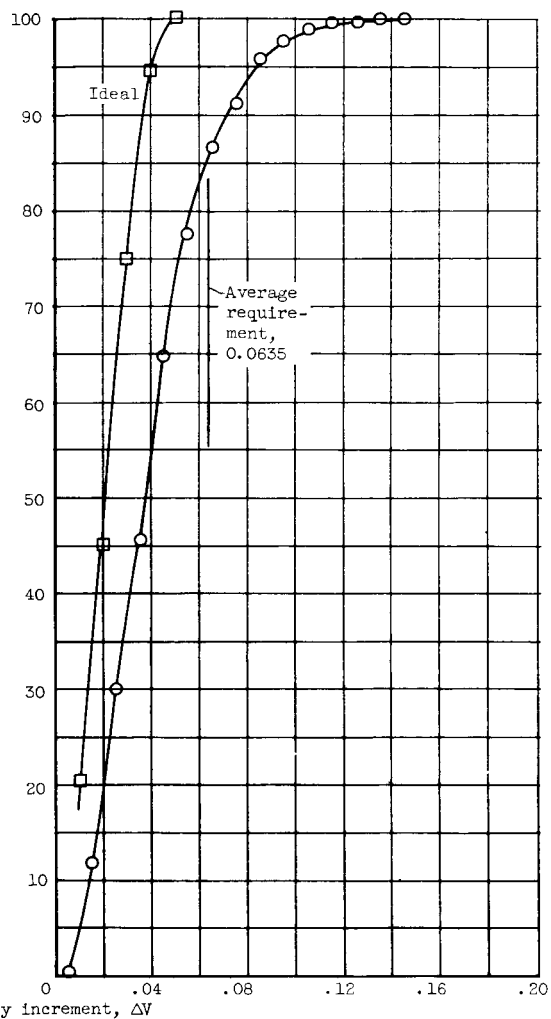


(c) Individual velocity-increment requirements.

Figure 5. - Continued. Results of conditional optimum solution.

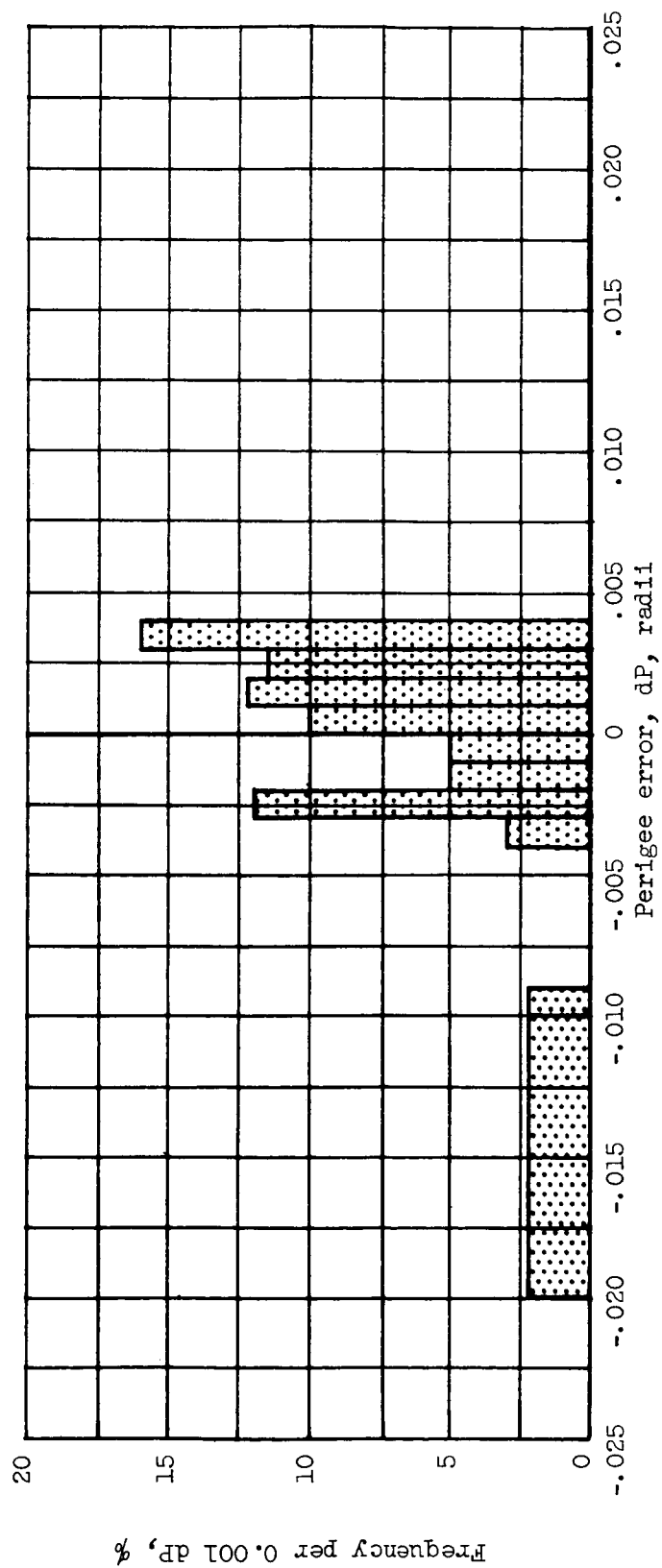


(d) Distribution of total velocity increments.



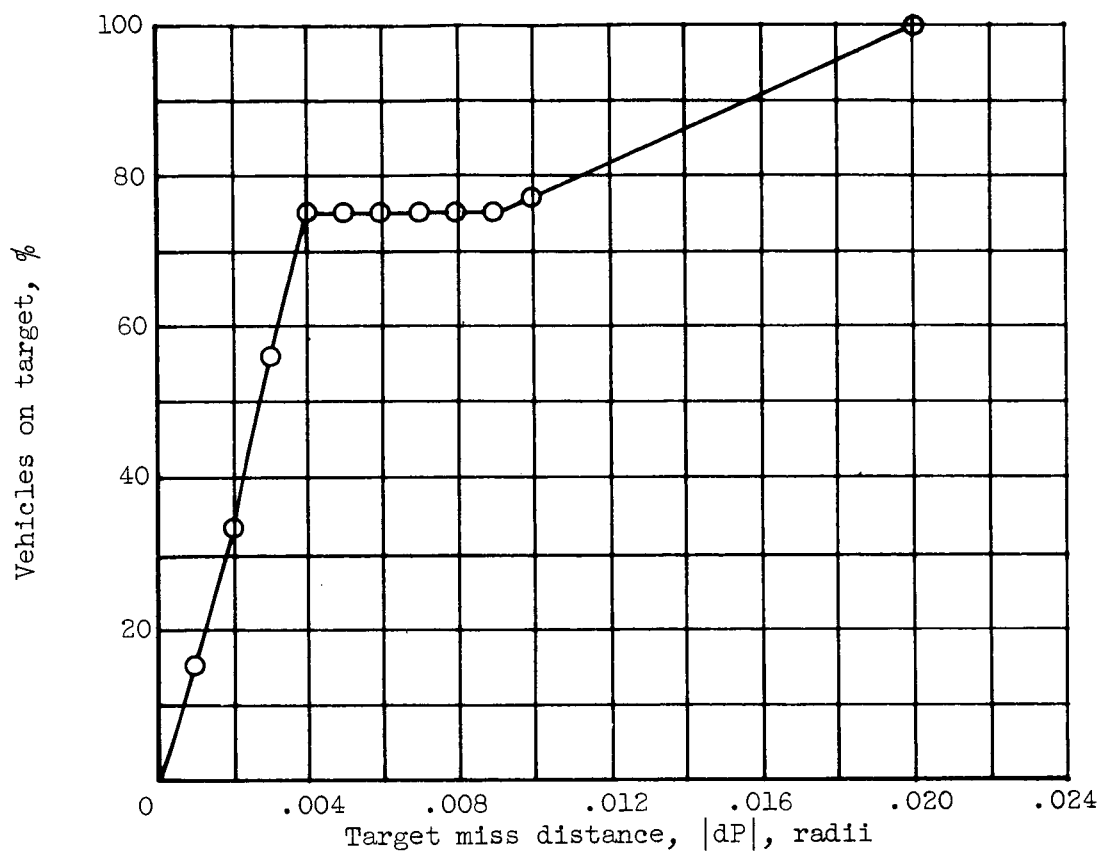
(e) Total velocity-increment requirements.

Figure 5. - Concluded. Results of conditional optimum solution.



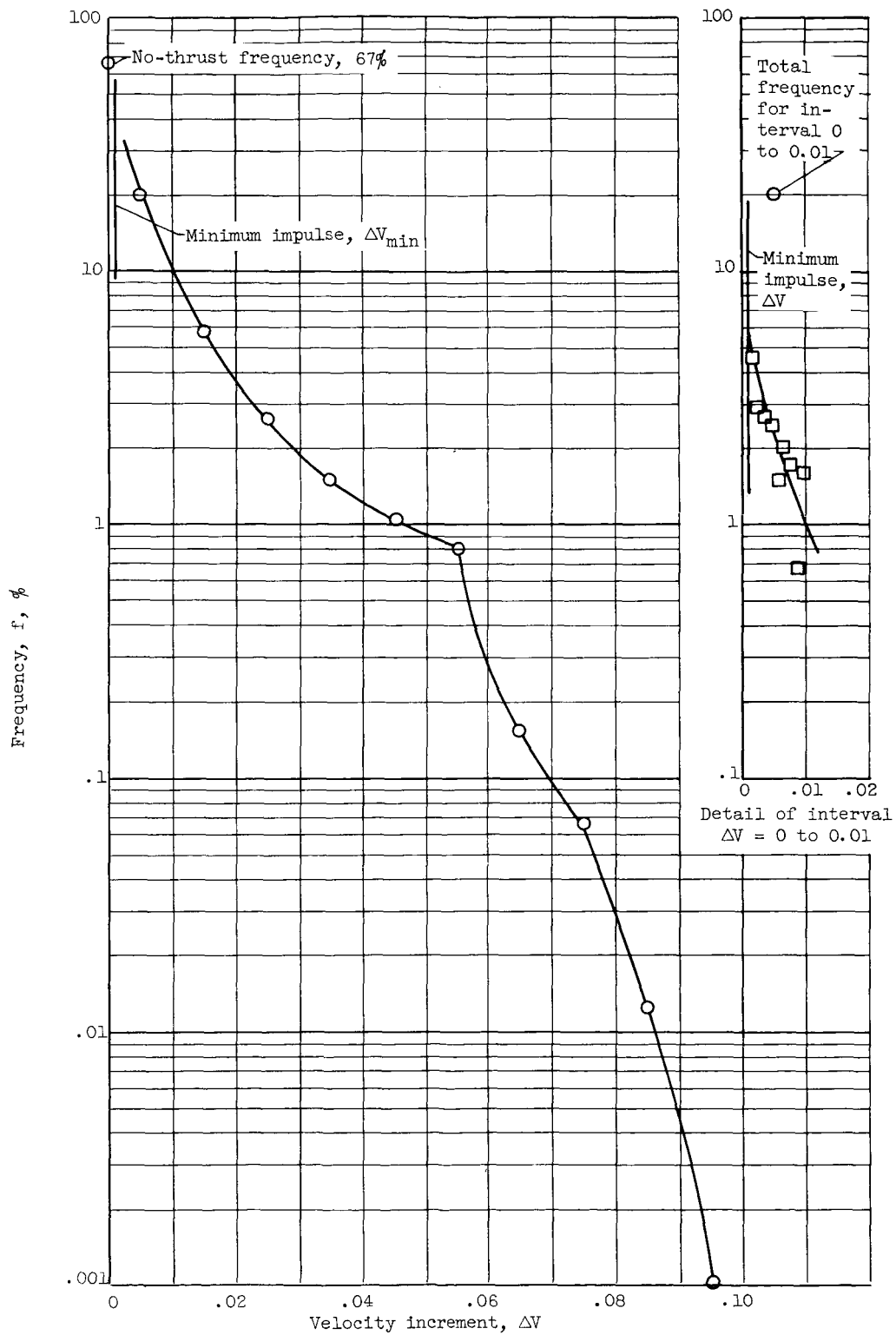
(a) Distribution of vehicles around target perigee after guidance.

Figure 6. - Results of reference solution.



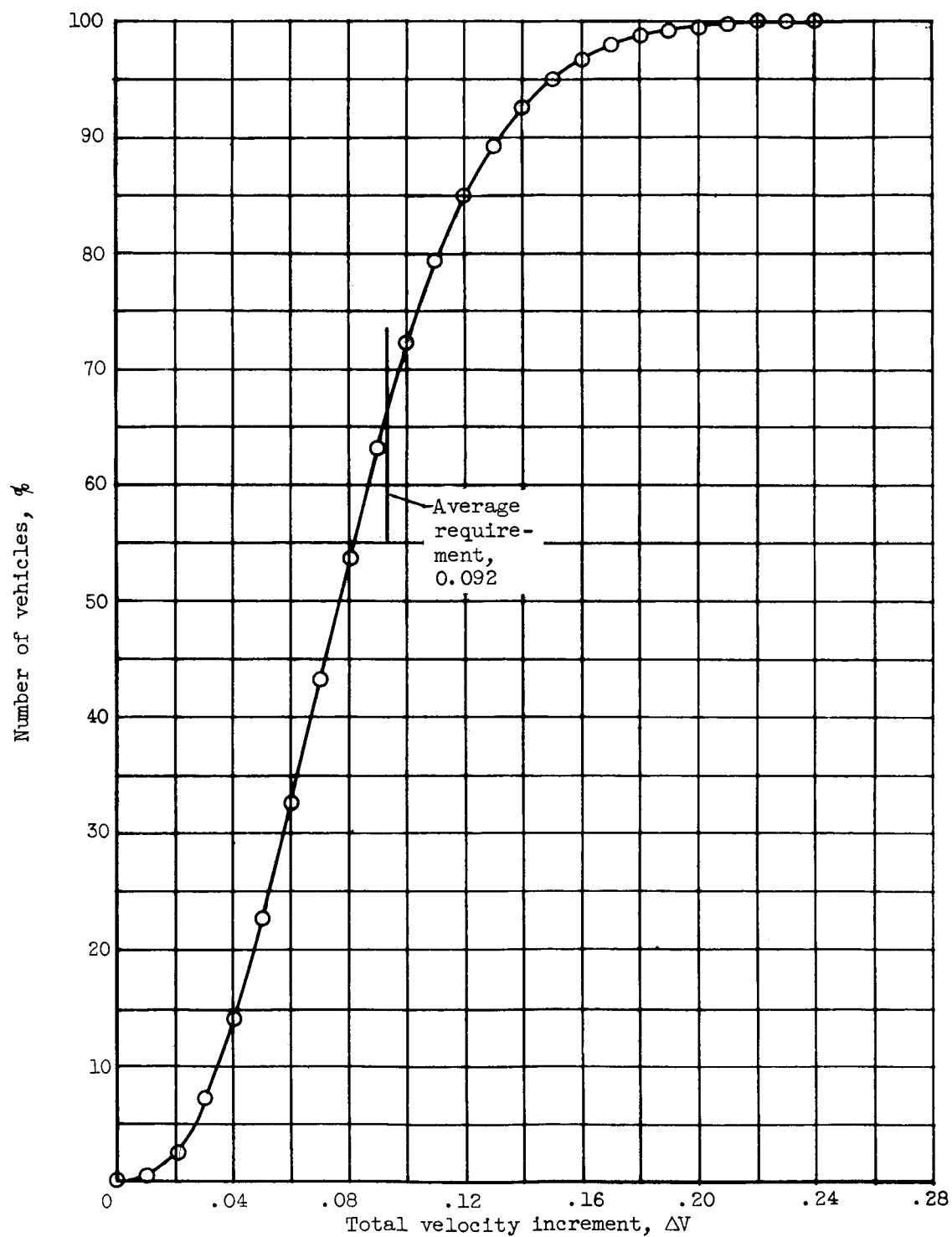
(b) Guidance accuracy.

Figure 6. - Continued. Results of reference solution.



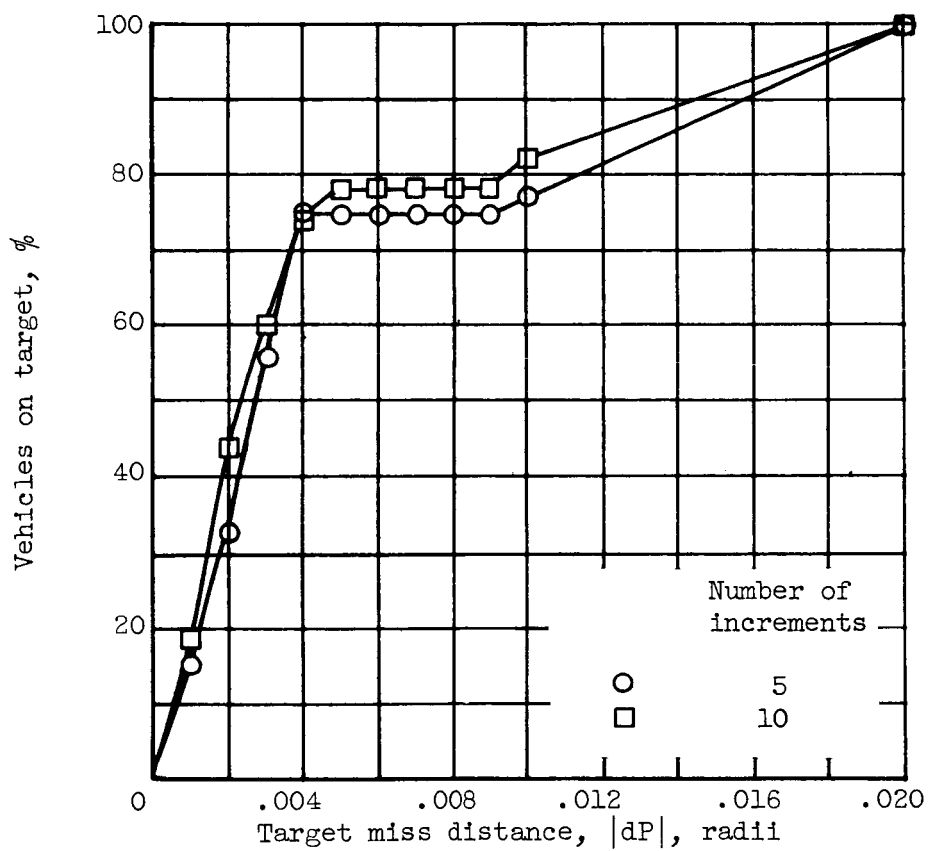
(c) Individual velocity-increment requirements.

Figure 6. - Continued. Results of reference solution.



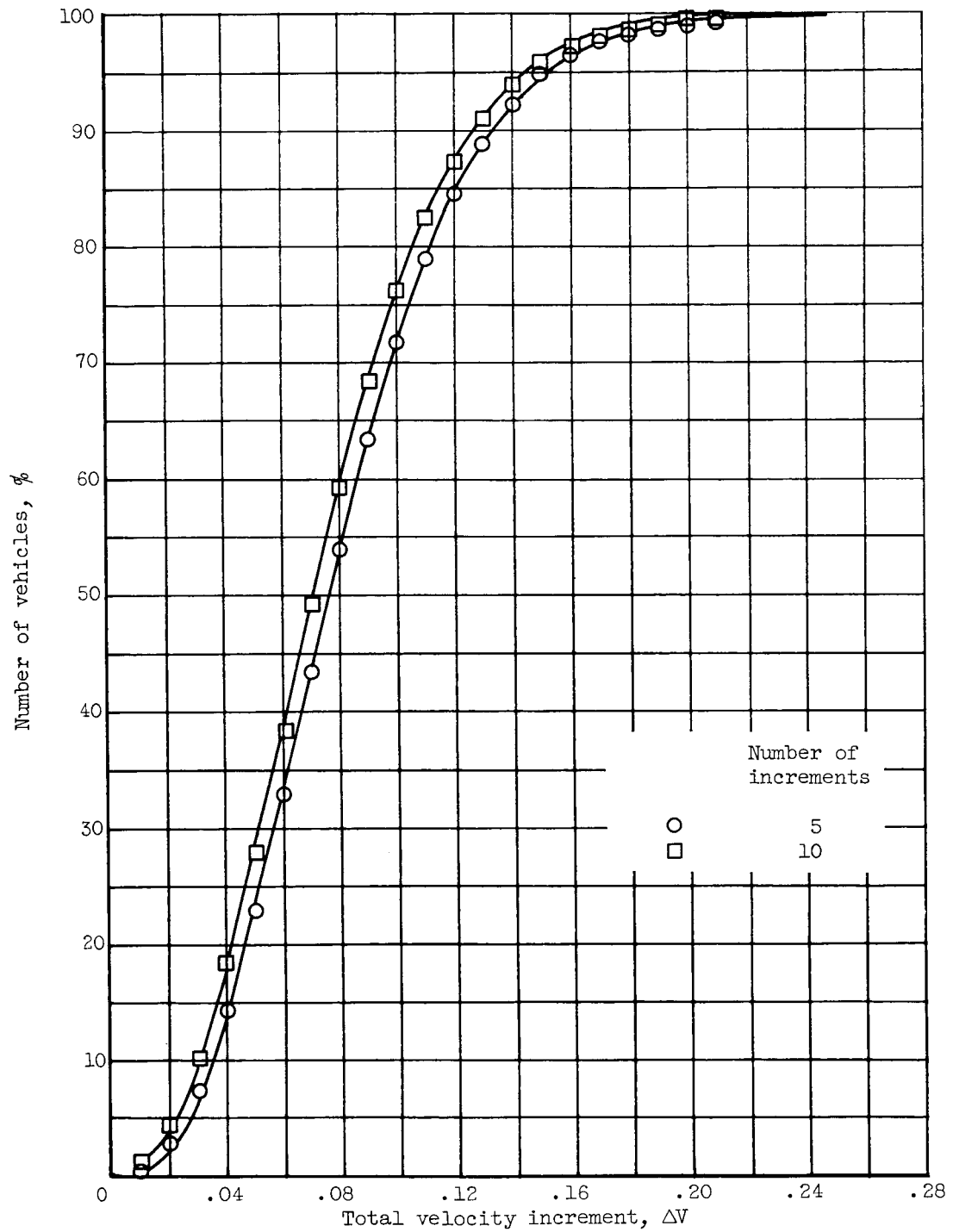
(d) Total velocity-increment requirements.

Figure 6. - Concluded. Results of reference solution.



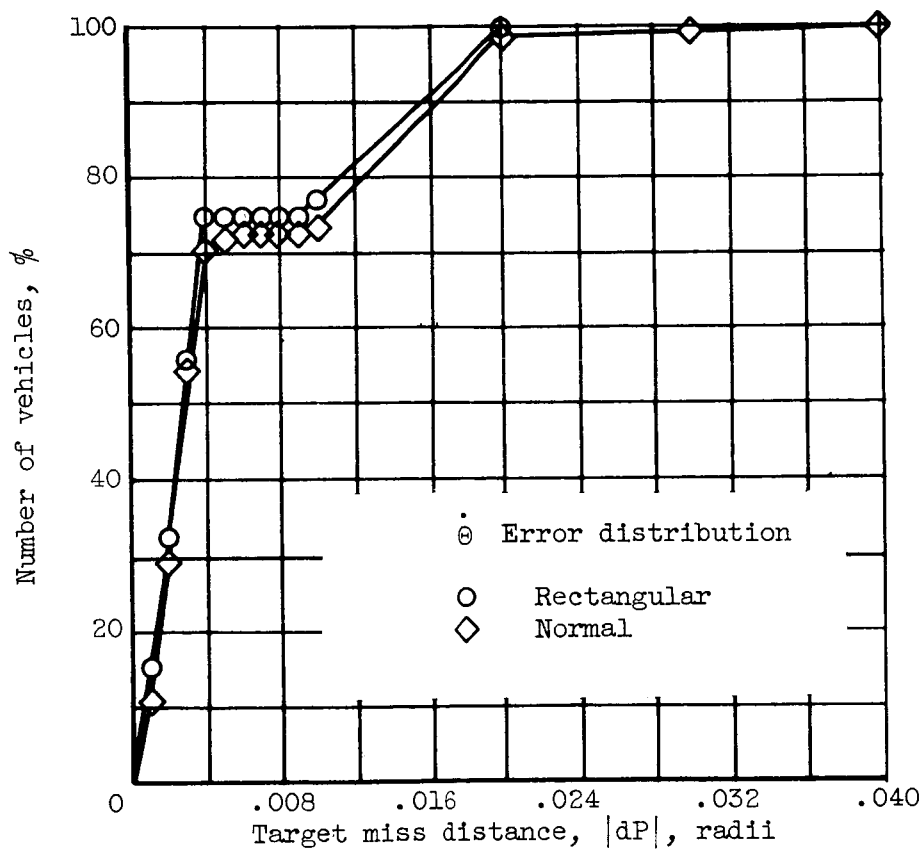
(a) Probability of vehicles' hitting target band of given size.

Figure 7. - Check of digital accuracy by variation in number of increments assumed in measurement error distributions, $\text{err}^{\hat{\theta}}$. Assumed values (table I).



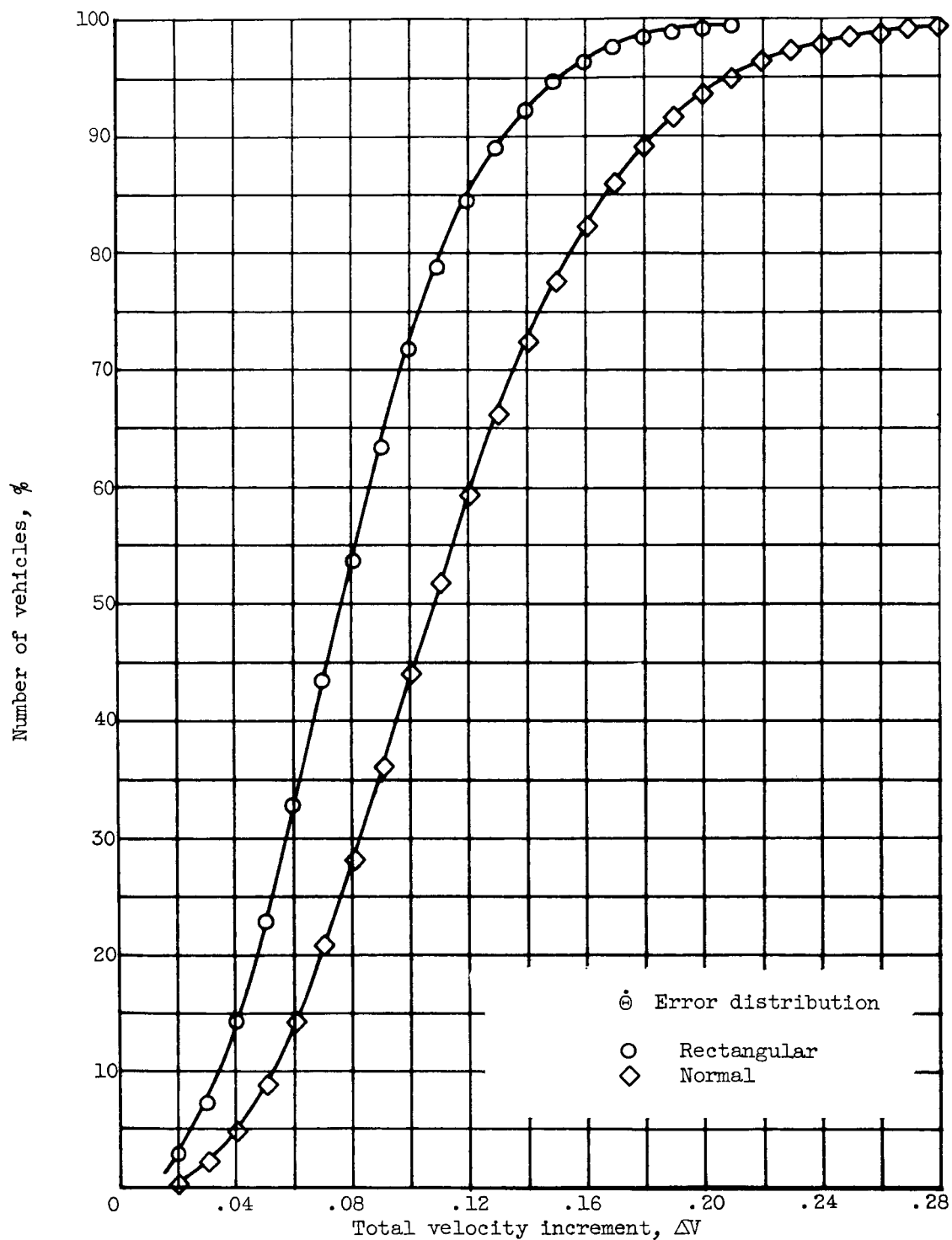
(b) Total velocity-increment requirements.

Figure 7. - Concluded. Check of digital accuracy by variation in number of increments assumed in measurement error distributions, $\text{err} \hat{\theta}$. Assumed values (table I).



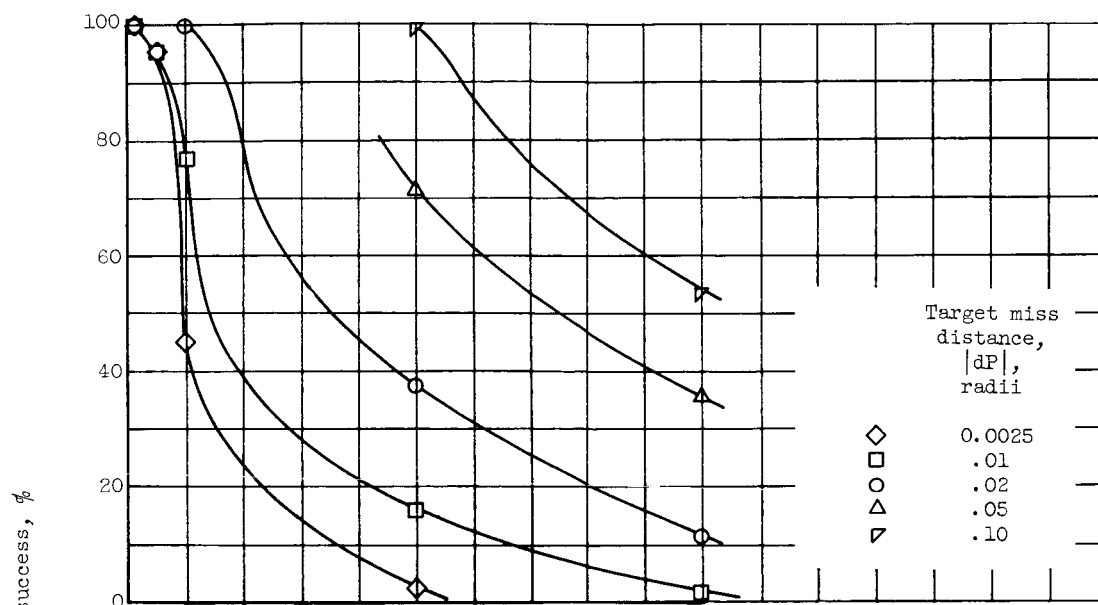
(a) Guidance accuracy.

Figure 8. - Effects of variation in shape of measurement error distribution. Assumed values (table I).

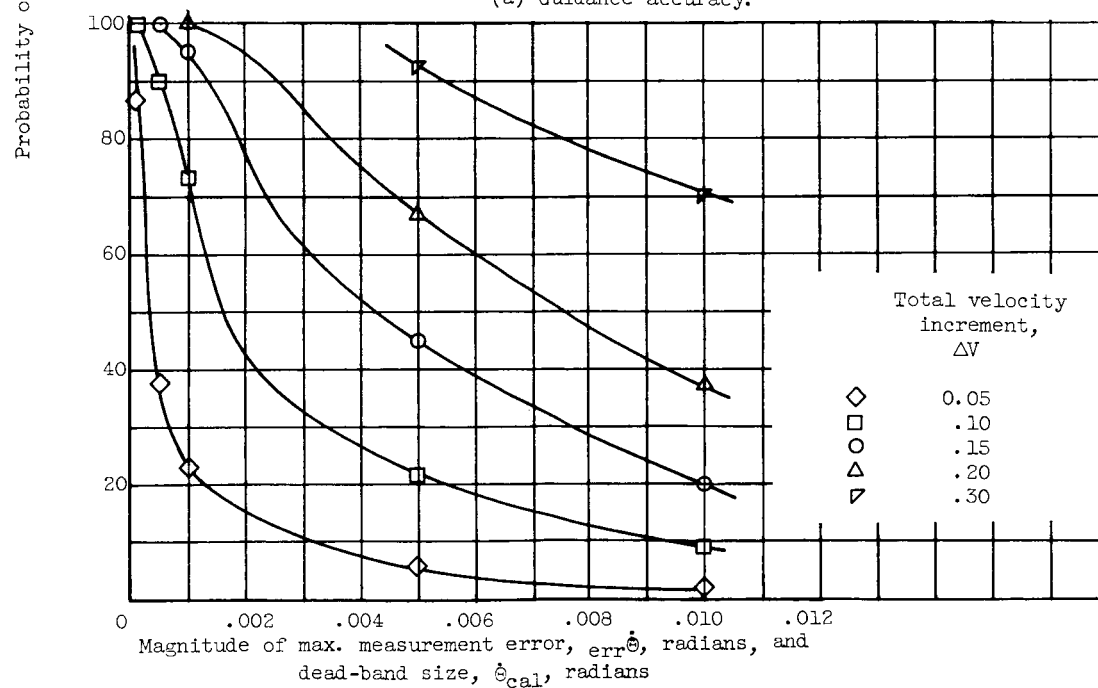


(b) Total velocity-increment requirements.

Figure 8. - Concluded. Effects of variation in shape of measurement error distribution. Assumed values (table I).

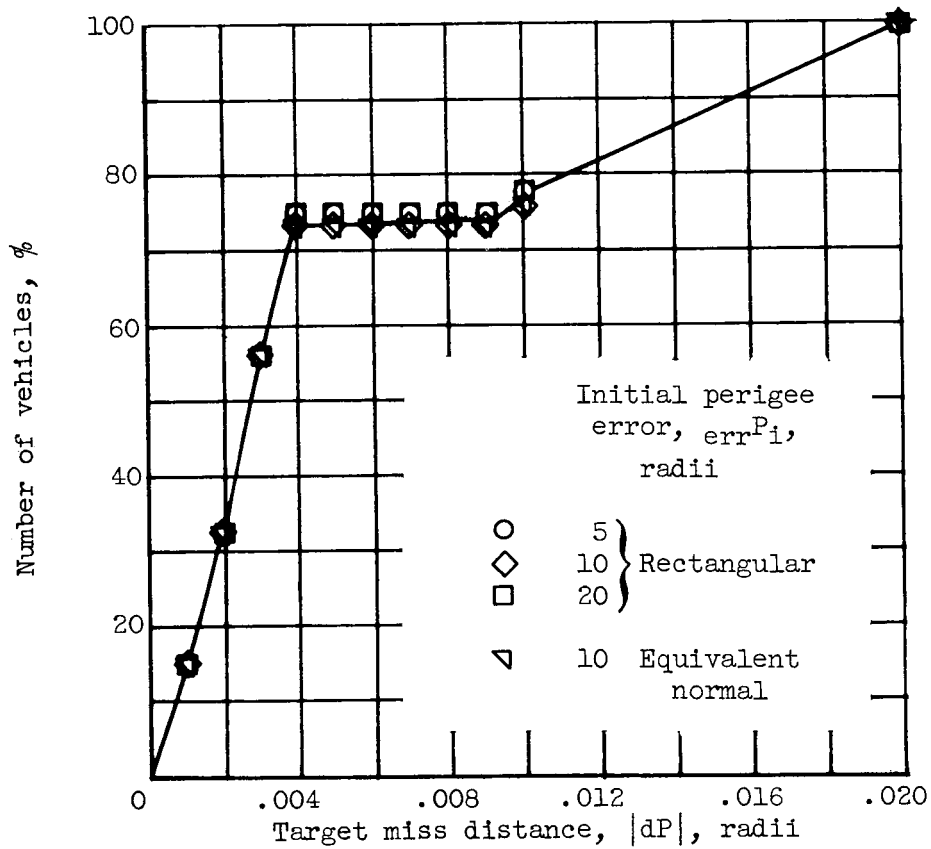


(a) Guidance accuracy.



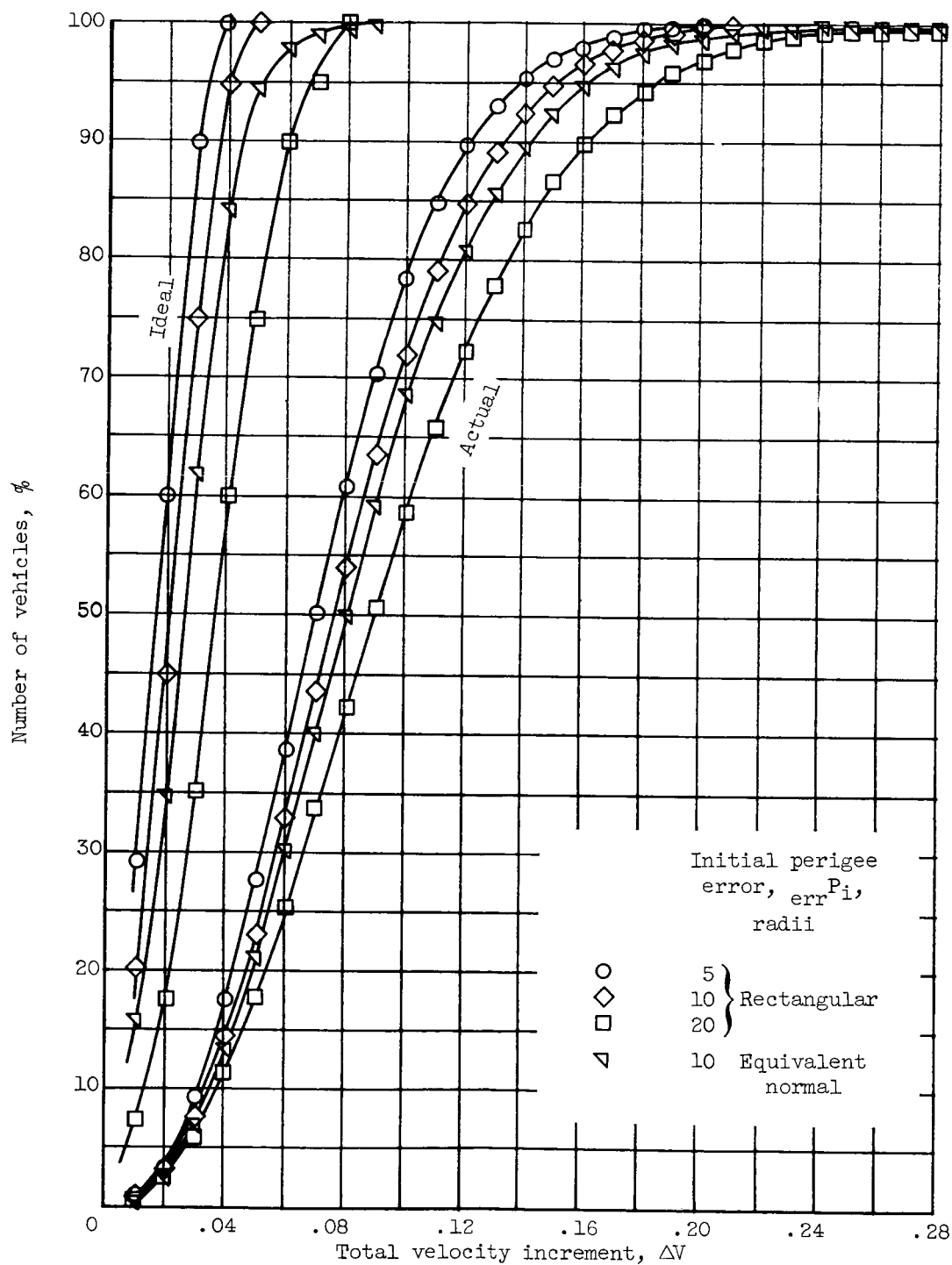
(b) Velocity-increment requirements.

Figure 9. - Effect of variation in measurement-error size. Dead-band size equal to maximum error; other assumed values from table I.



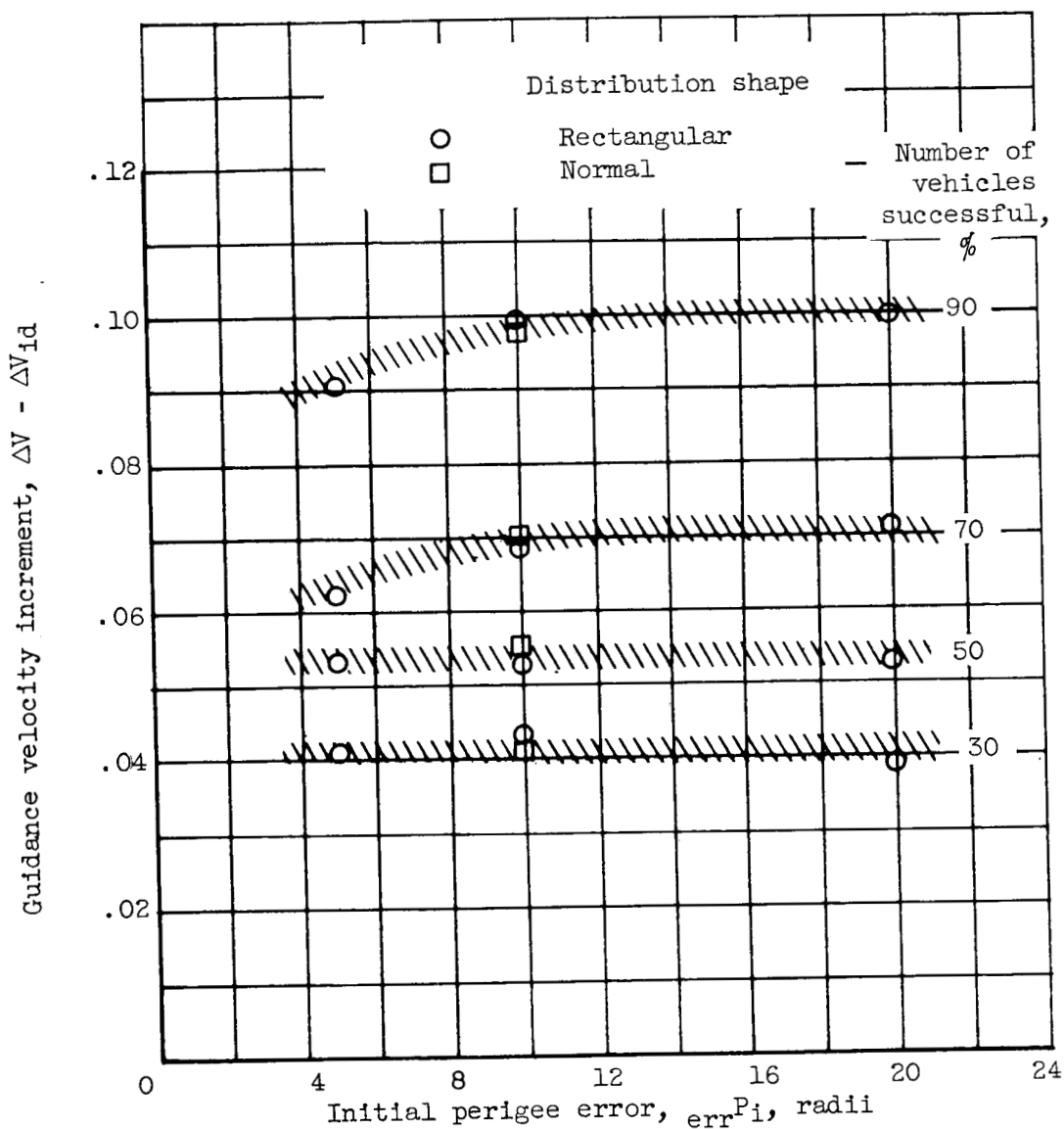
(a) Guidance accuracy.

Figure 10. - Effect of variation in initial perigee error assumptions. Assumed values (table I).



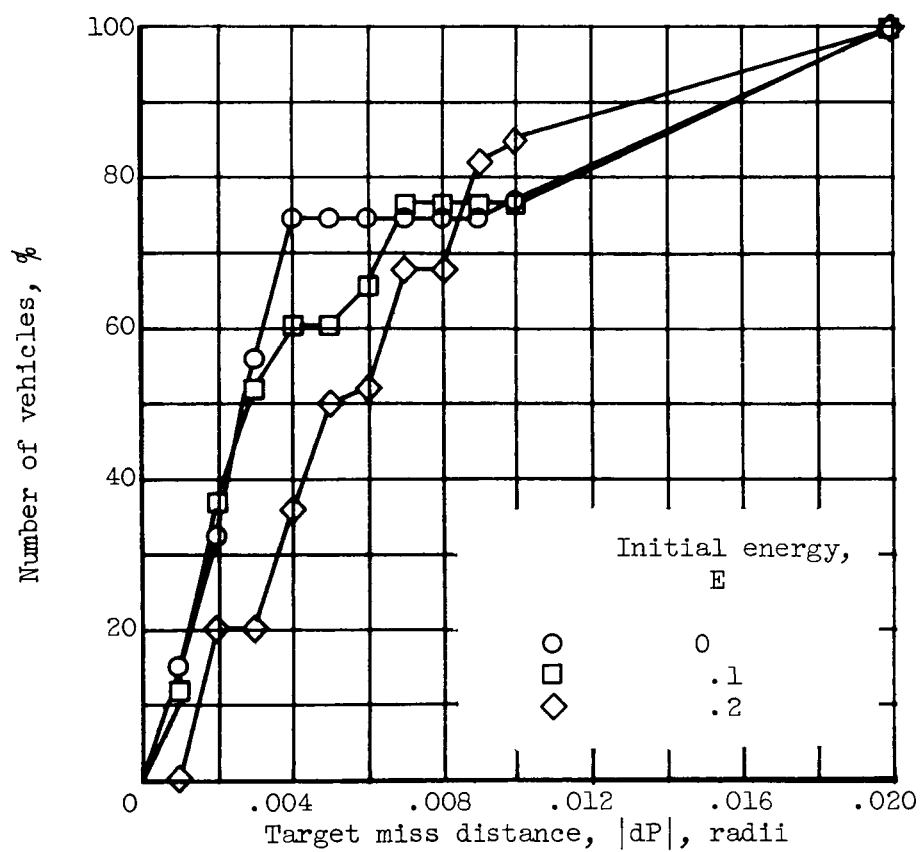
(b) Total velocity-increment requirements.

Figure 10. - Continued. Effect of variation in initial perigee error assumptions. Assumed values (table I).



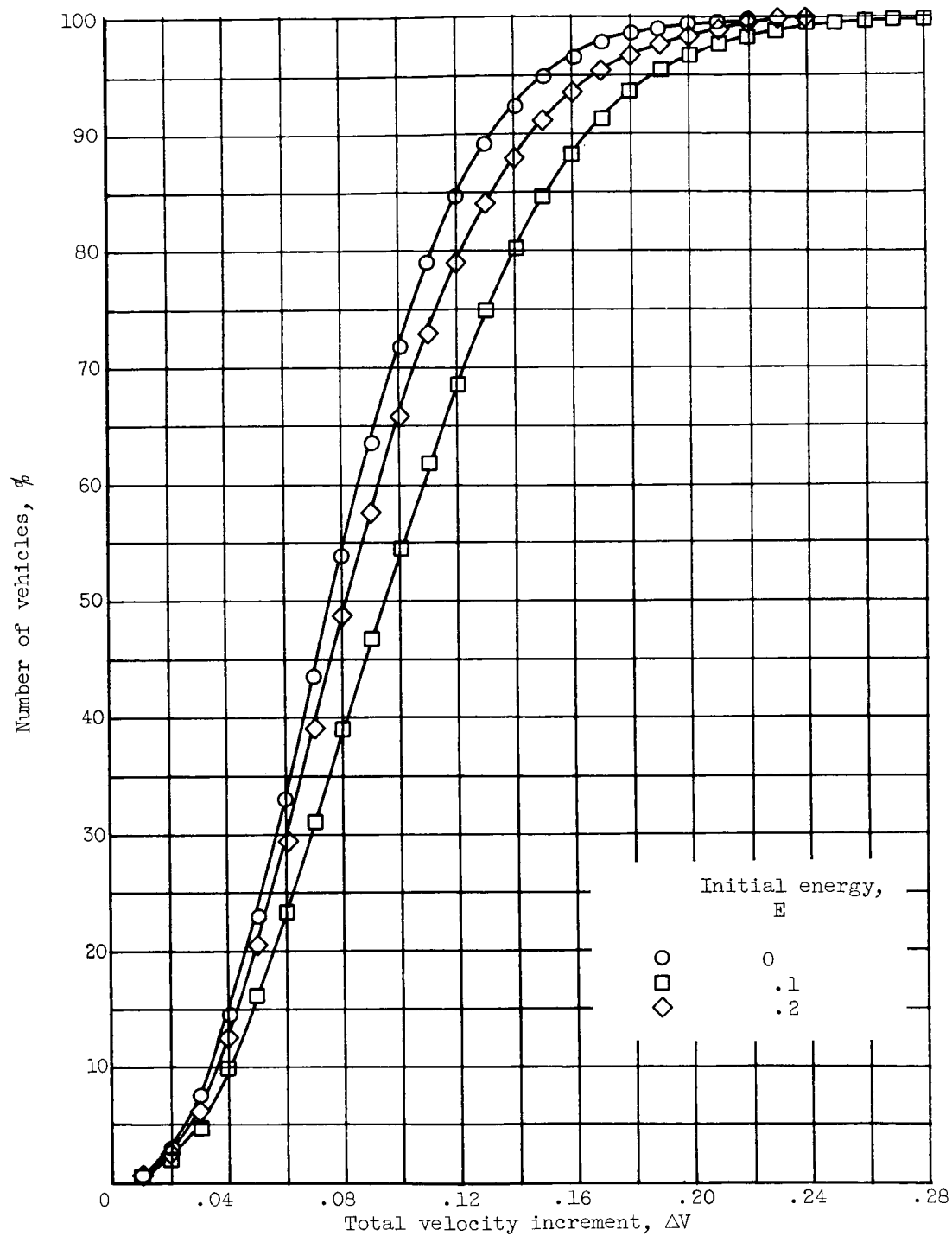
(c) Guidance velocity increment.

Figure 10. - Concluded. Effect of variation in initial perigee error assumptions. Assumed values (table I).



(a) Guidance accuracy.

Figure 11. - Effects of variation in initial energy E. Assumed values (table I).



(b) Total velocity-increment requirement.

Figure 11. - Concluded. Effects of variation in initial energy E .
Assumed values (table I).

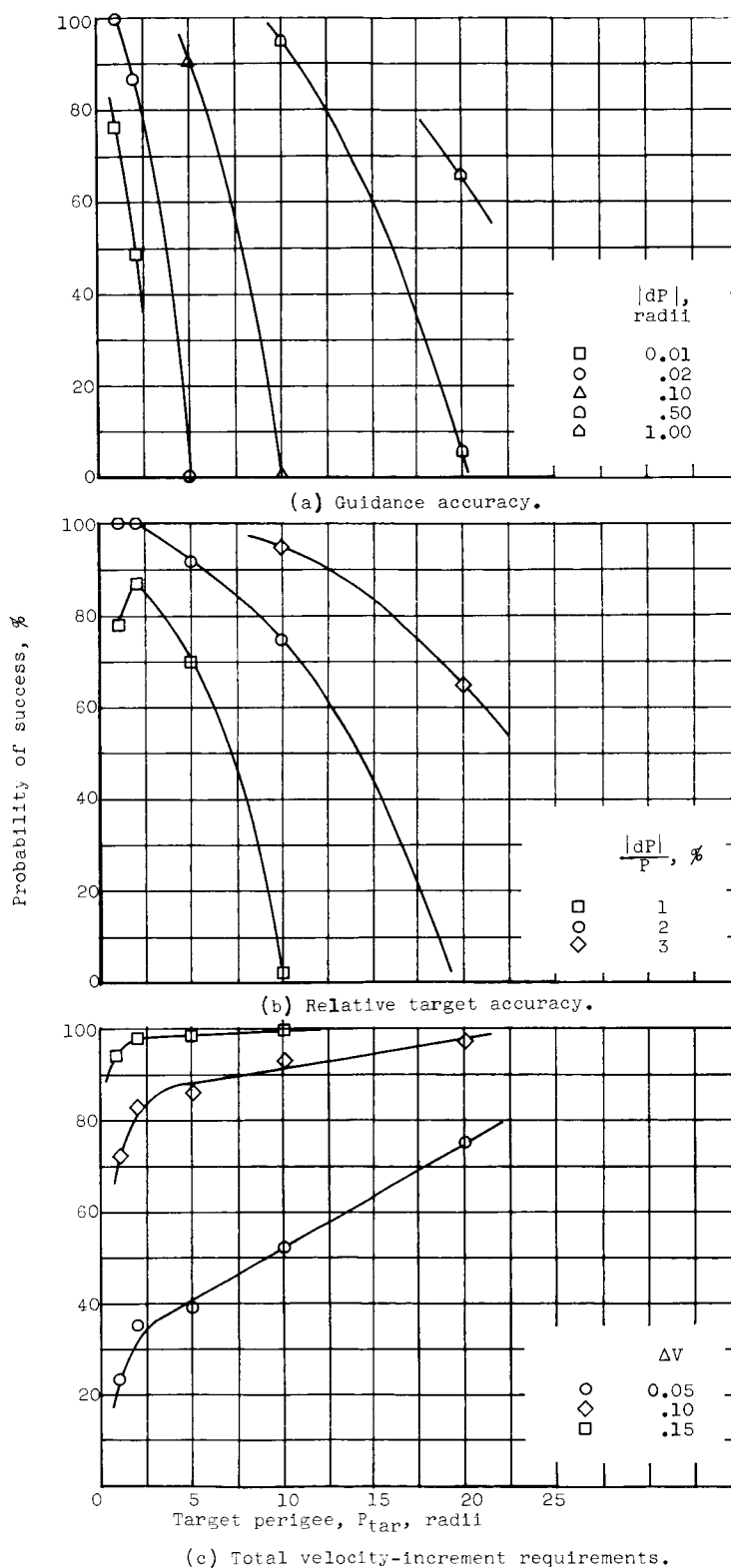
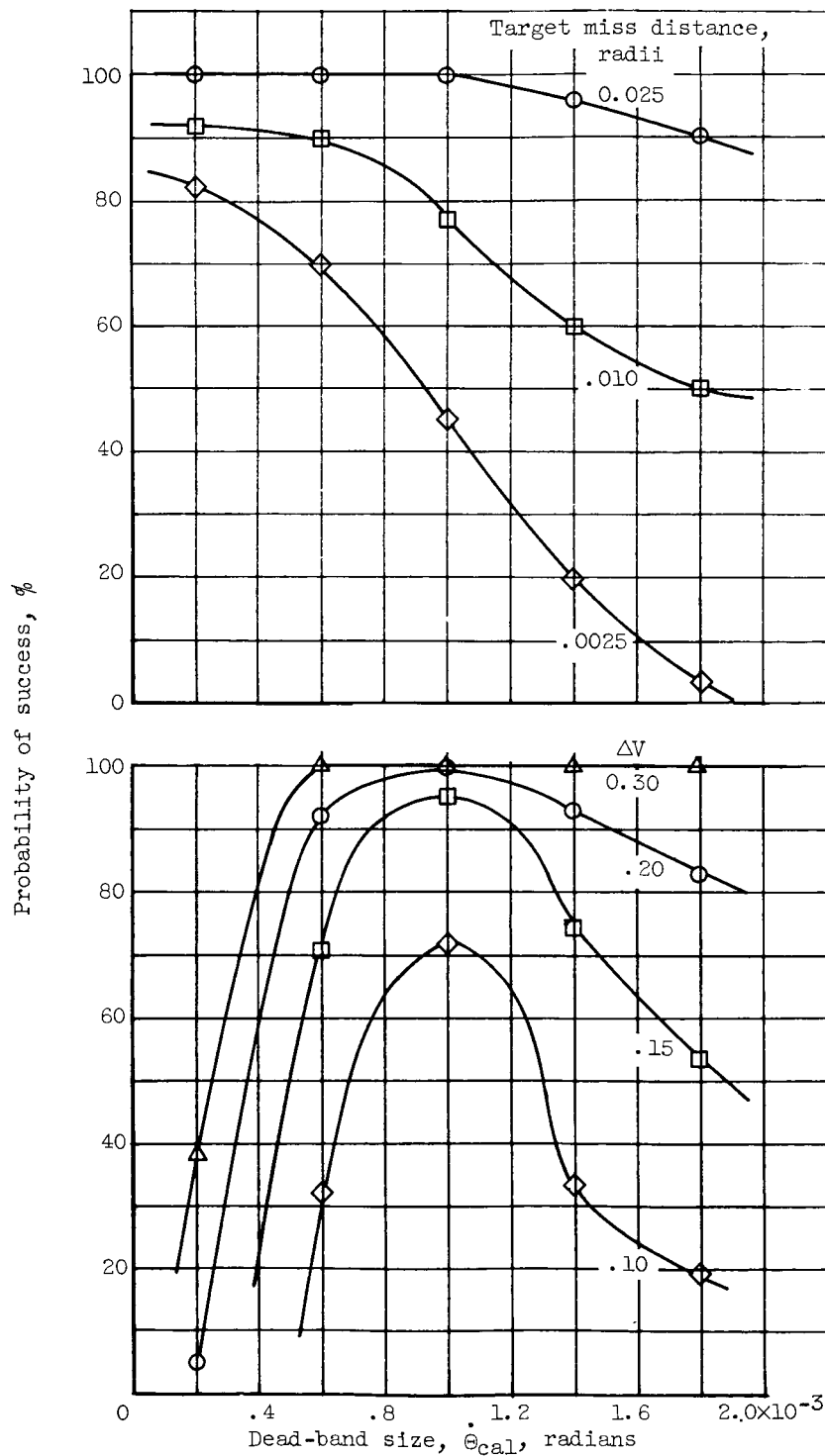
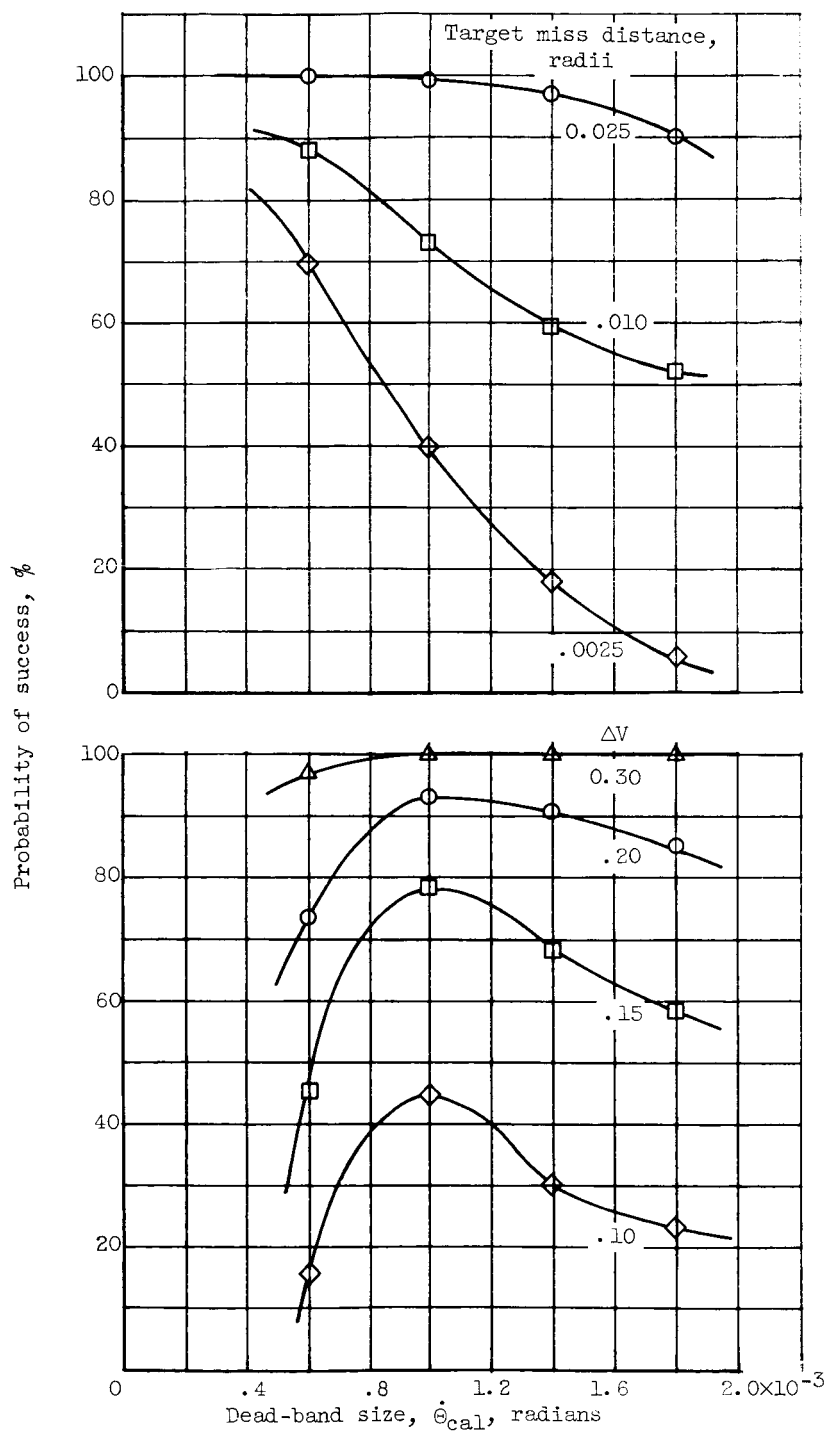


Figure 12. - Effects of variation in target perigee P_{tar} .
Assumed values (table I).



(a) Summary of effects on guidance. Assumed values (table I).

Figure 13. - Effect of variation in dead-band size.



(b) Summary of effects on guidance. Measurement-error distribution: Standard deviation with expected error of $\pm 0.5 \times 10^{-3}$ radian. Other assumed values (table I).

Figure 13. - Concluded. Effect of variation in dead-band size.

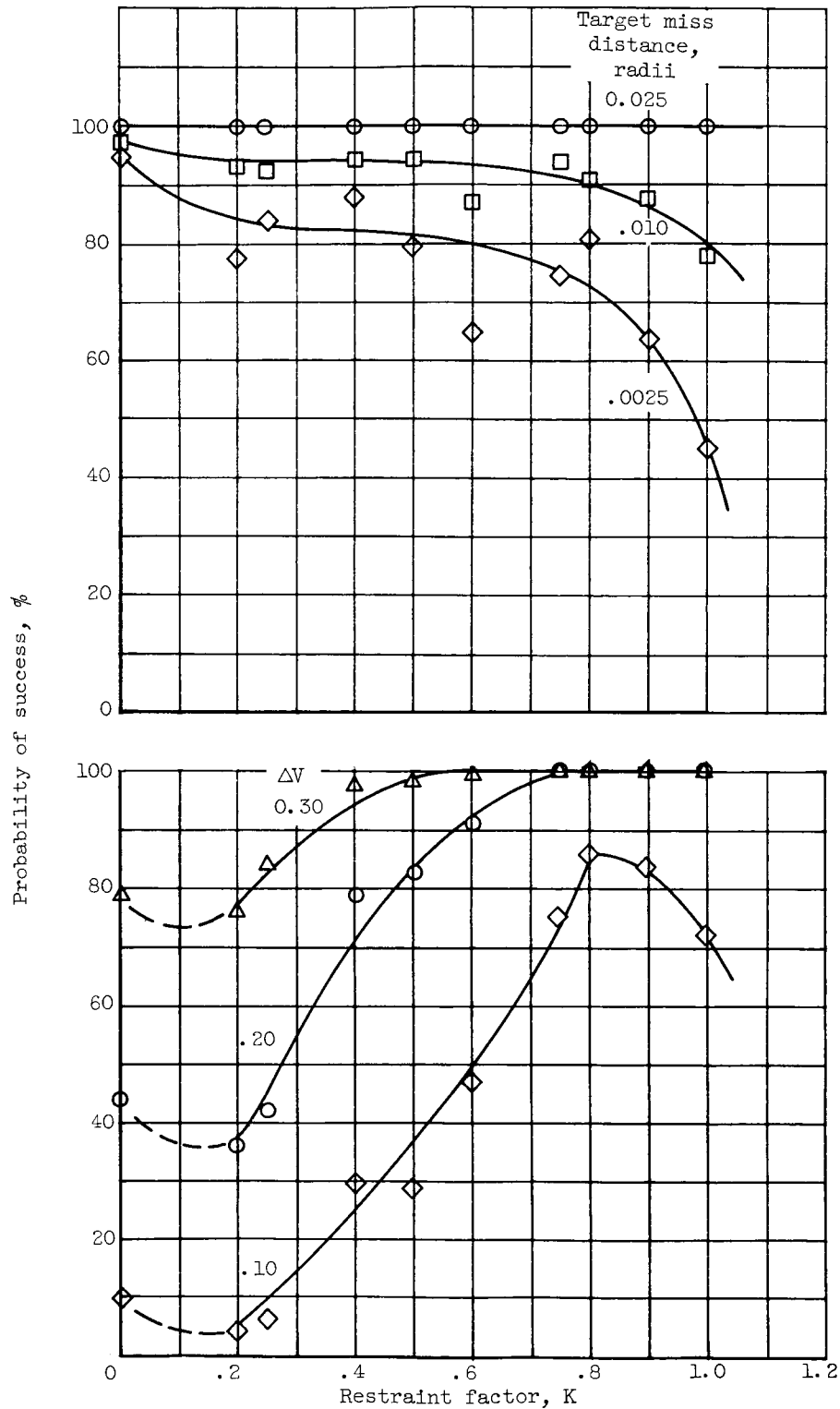


Figure 14. - Effect of variation in guidance restraint factor K. Assumed values (table I).

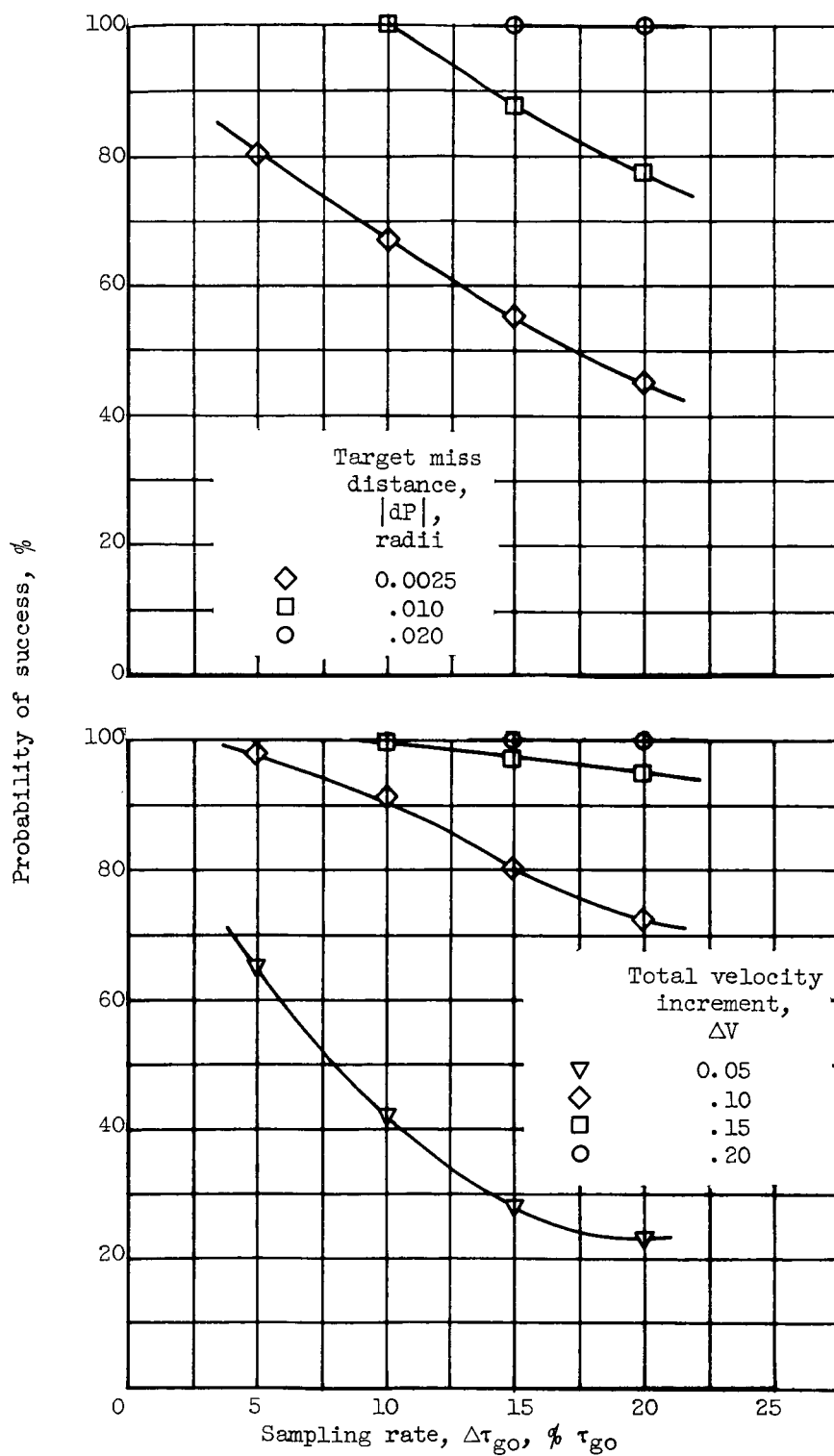


Figure 15. - Effects of variation in sampling rate $\Delta\tau_{go}$. Assumed values (table I).

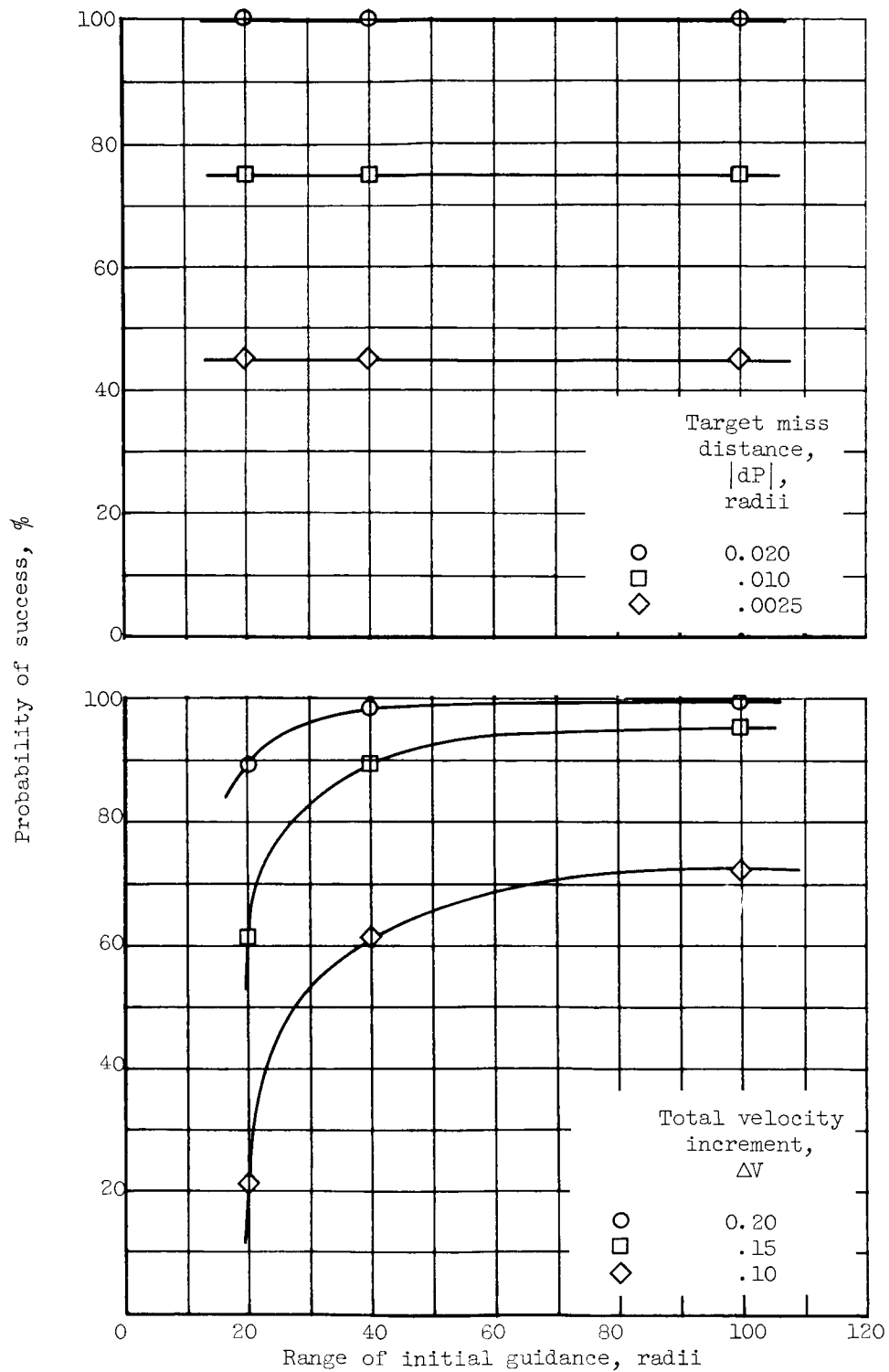


Figure 16. - Effect of variation in range of initial guidance. Assumed values (table I).

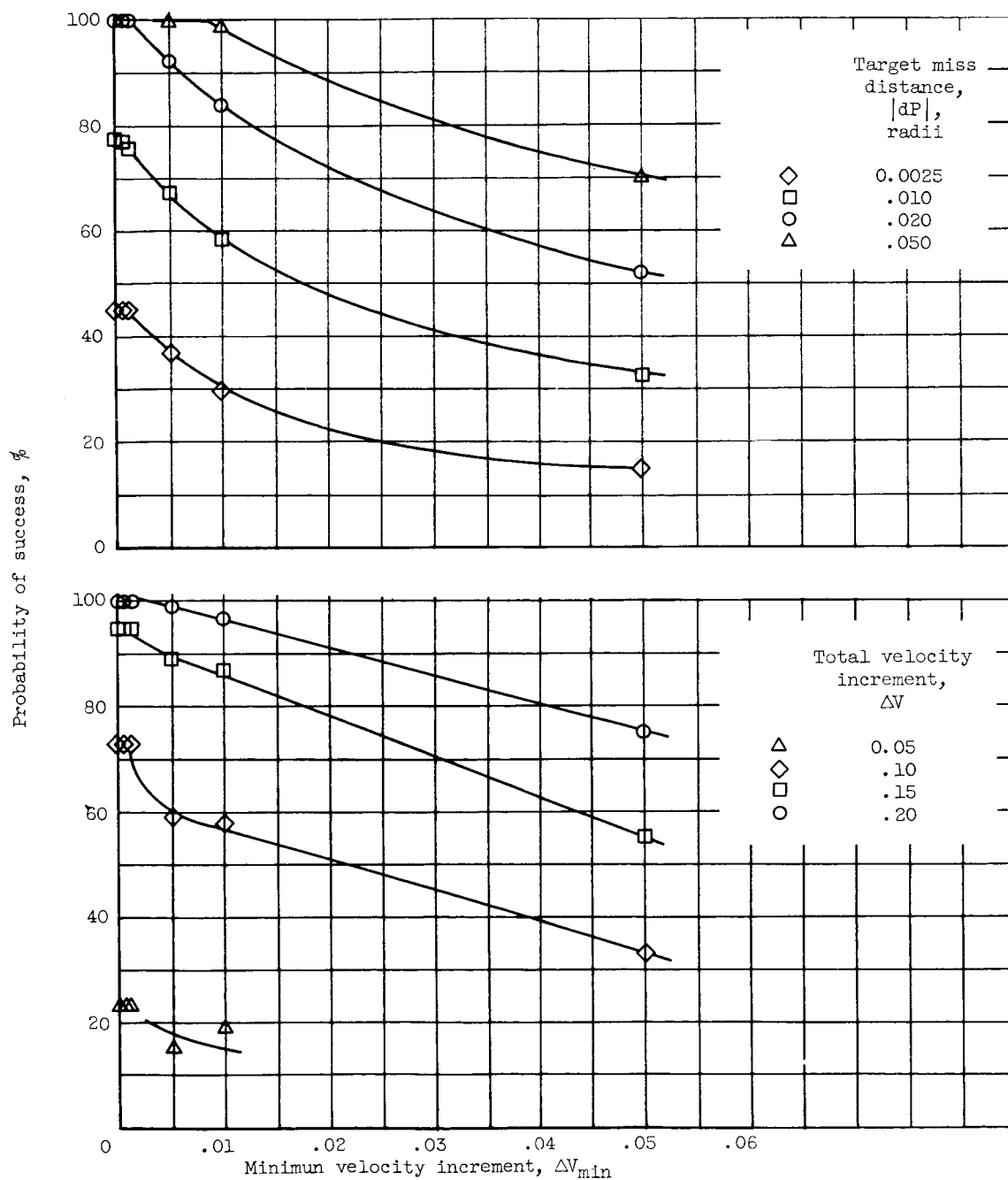


Figure 17. - Effect of limiting minimum velocity increment ΔV_{min} . Assumed values (table I).

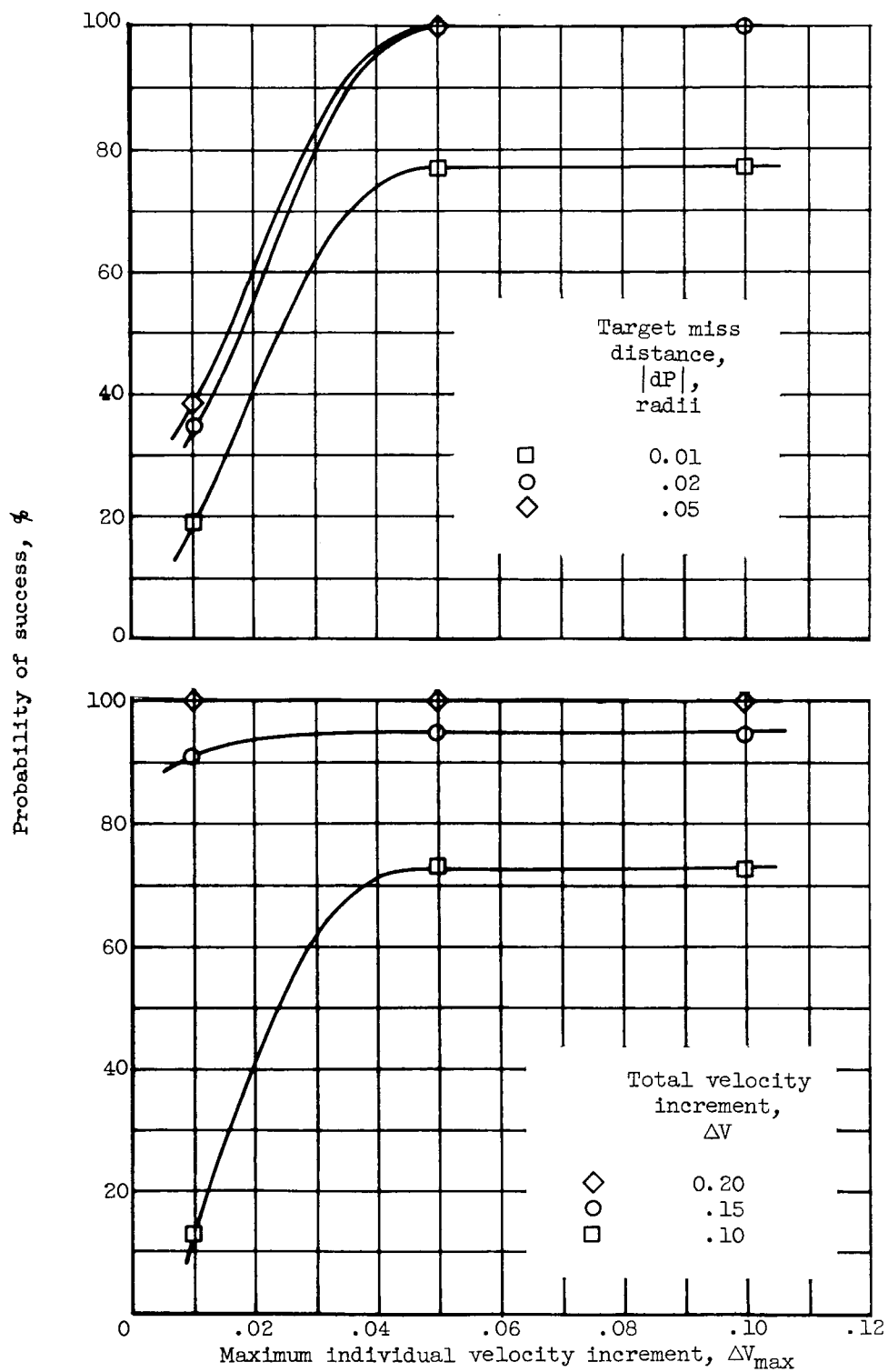


Figure 18. - Effect of limiting maximum velocity increment $\Delta V_{\max}$. Assumed values (table I).

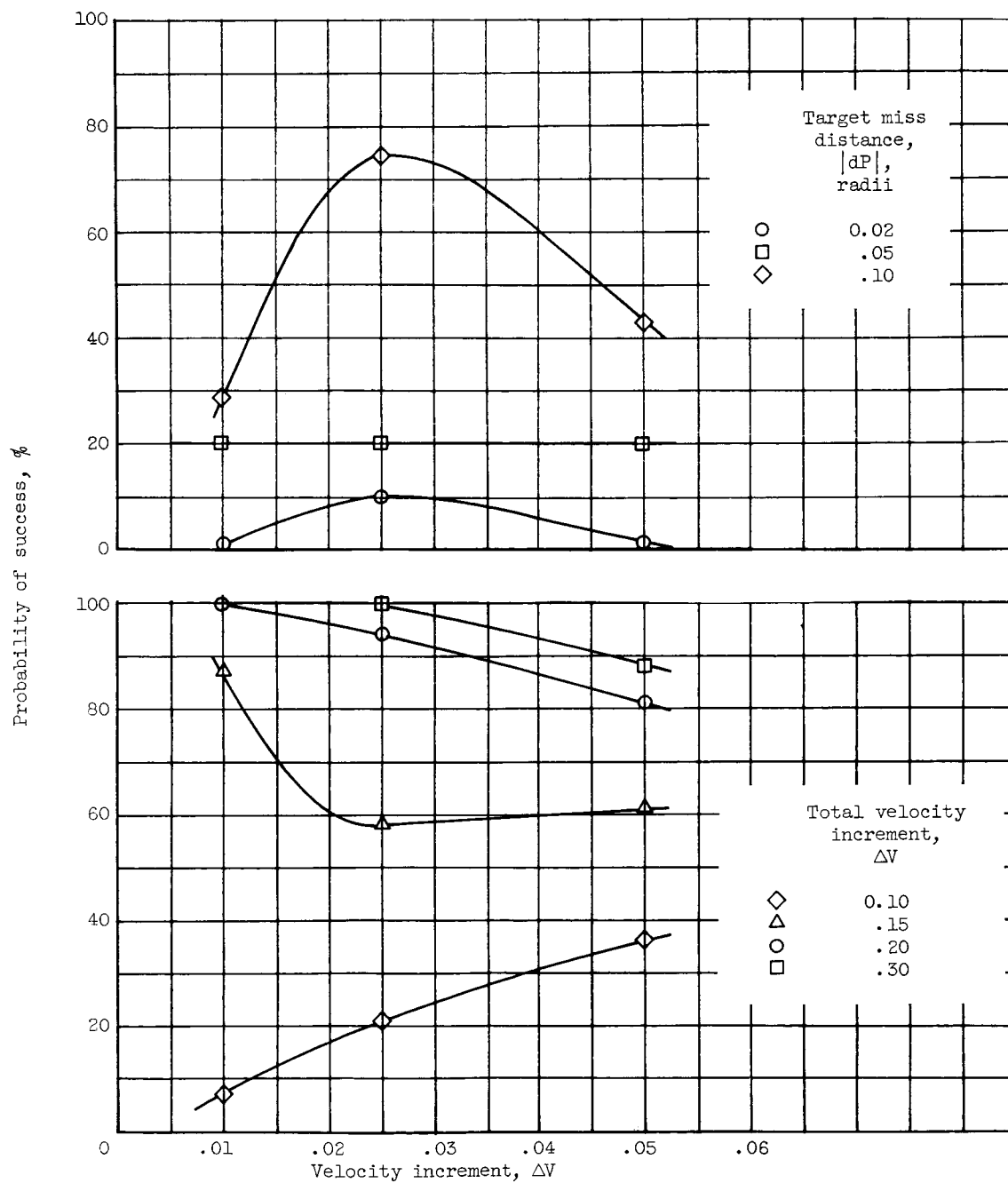


Figure 19. - Effects of using fixed-velocity-increment thrust devices. Assumed values (table I).

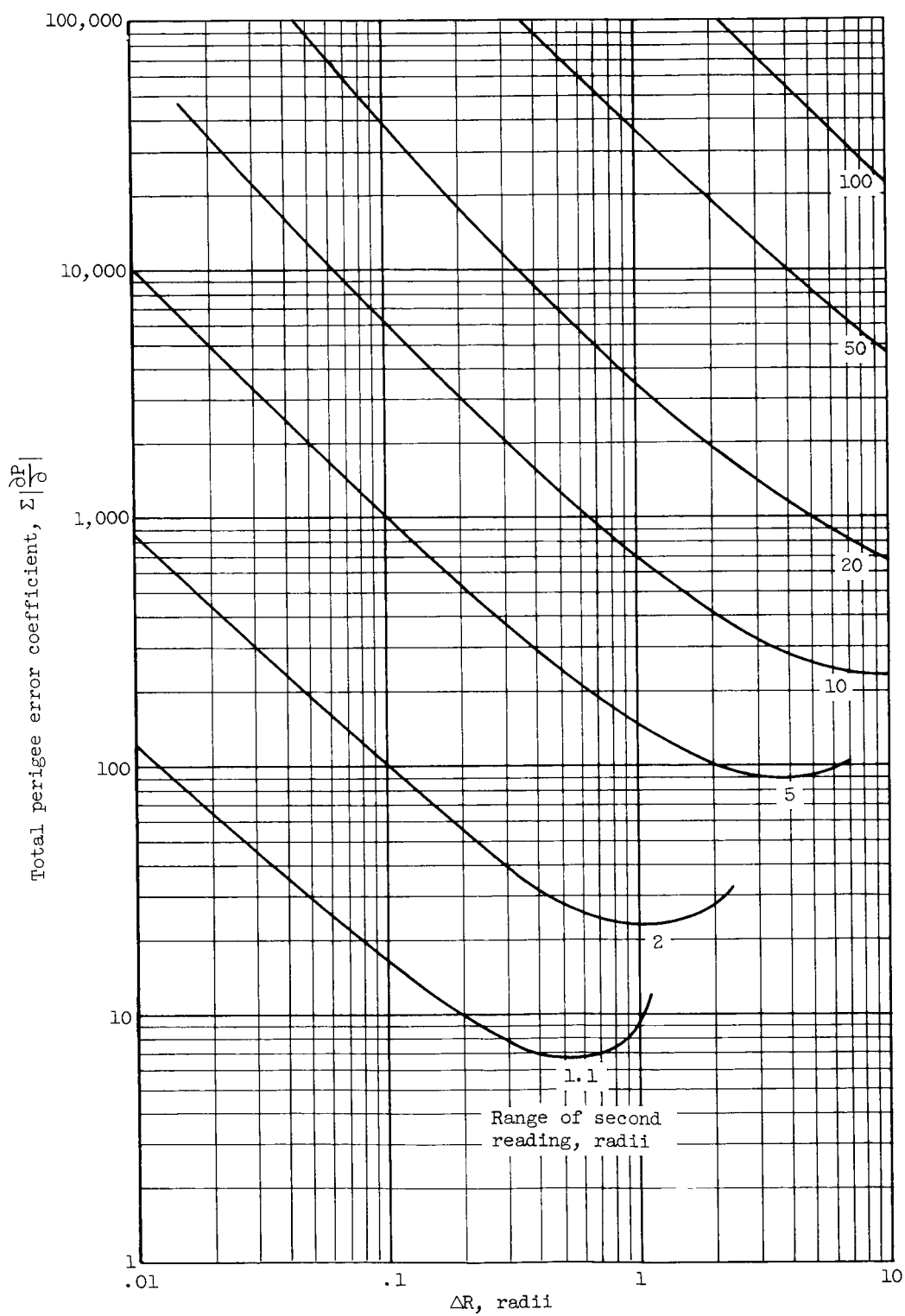


Figure 20. - Perigee error for $R, \dot{R}$ measurement scheme evaluated for $dR = d\dot{R}$, $E = 0$, and $P = 1$.

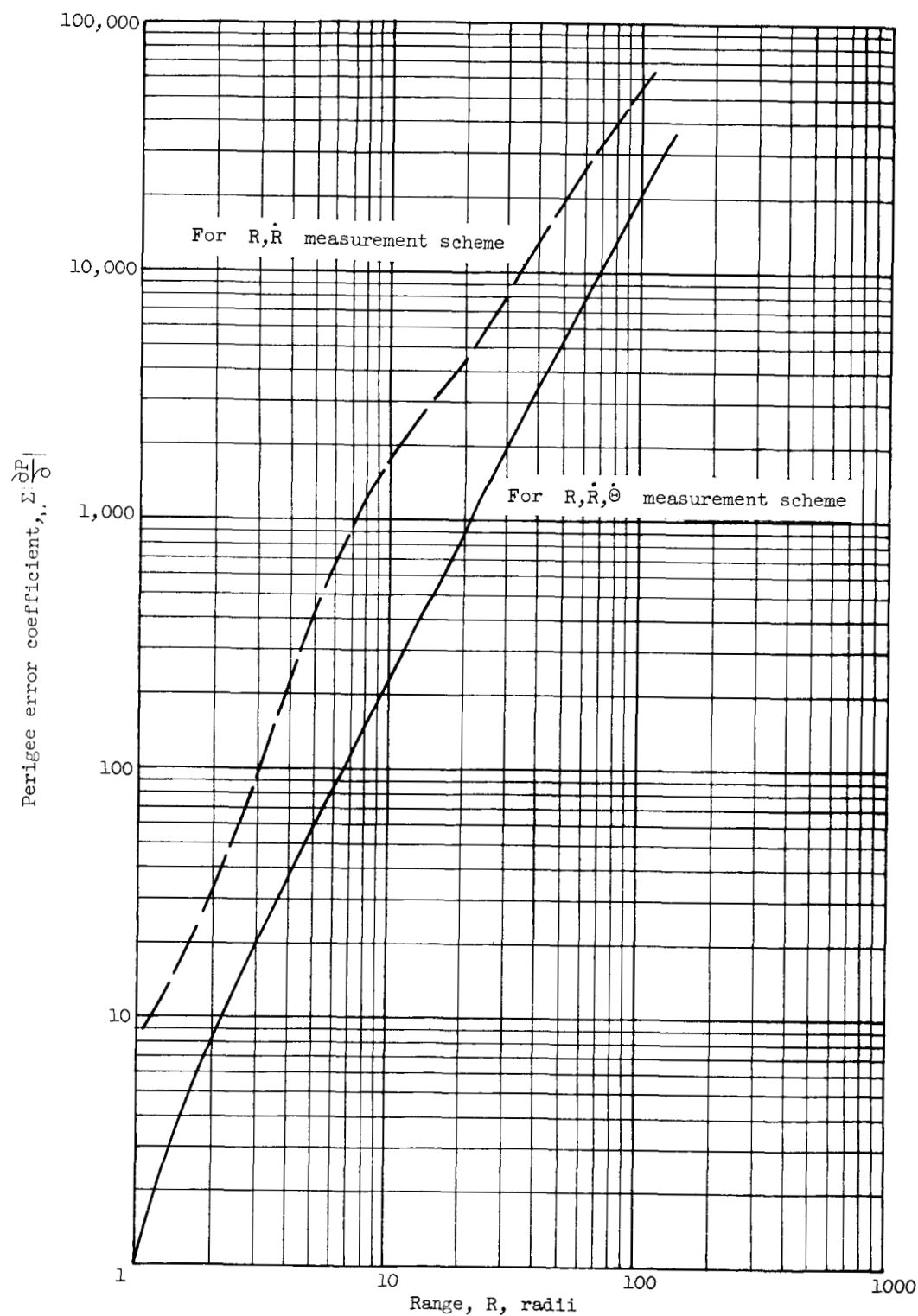


Figure 21: - Comparison of perigee error coefficient change with range for $R, \dot{R}$ scheme and $R, \dot{R}, \dot{\theta}$ scheme. Assumed sampling rate, $\Delta\tau_{go} = 5$ percent; $E = 0$, $P = 1$.

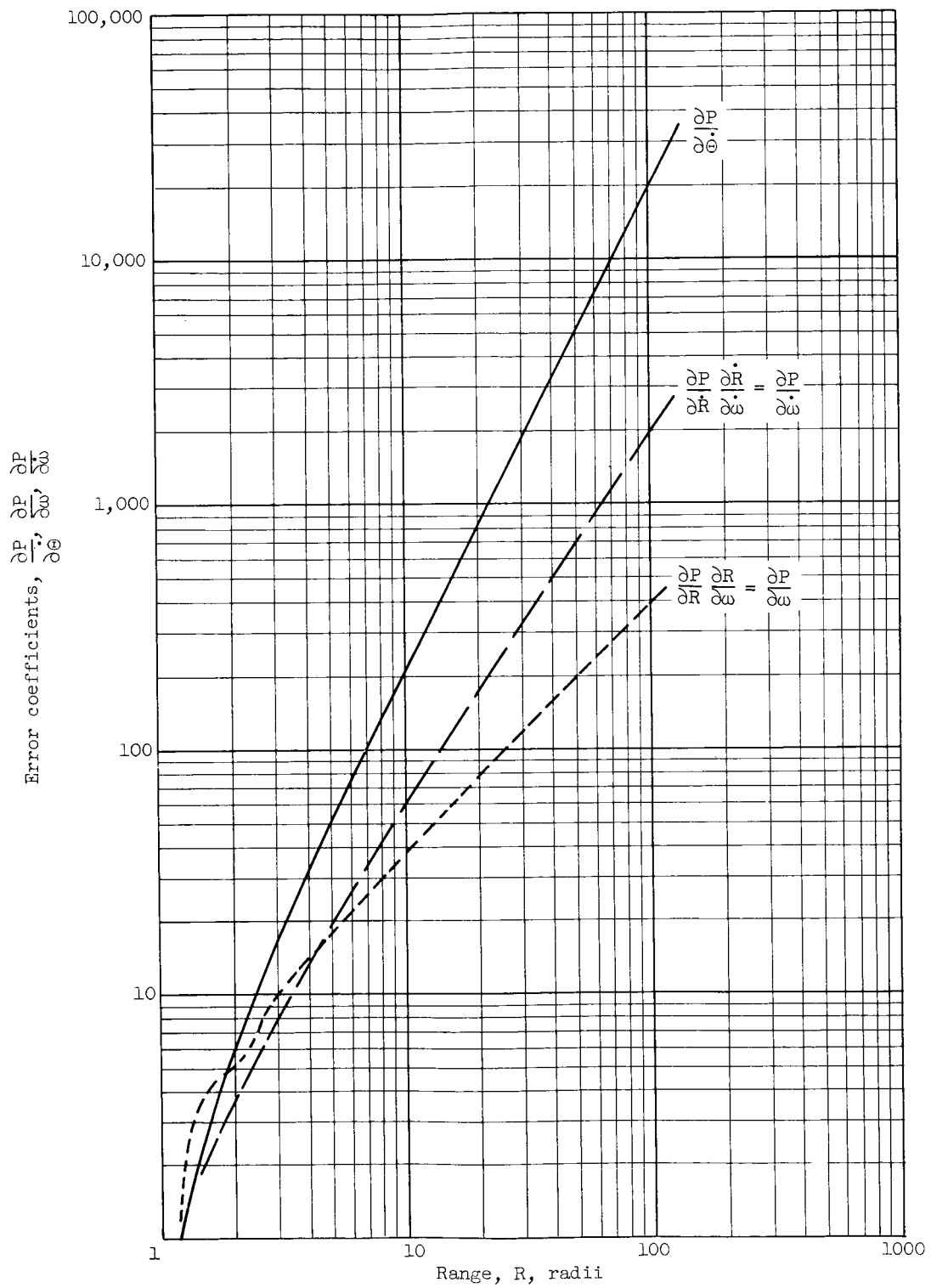
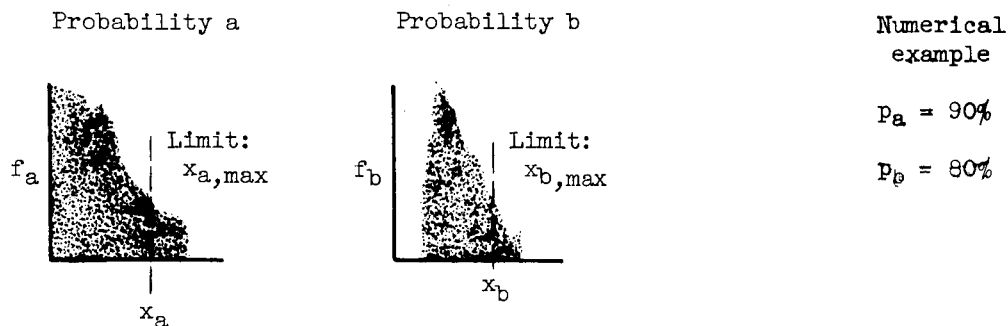


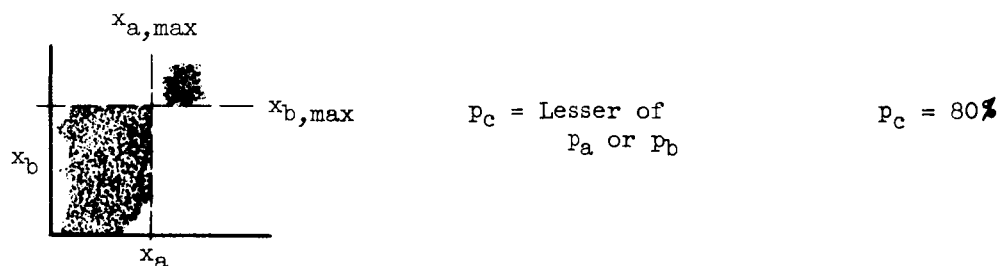
Figure 22. - Error coefficients of $R, \omega, \dot{\omega}$ measurements evaluated at $P = 1, E = 0$.

For two frequency distributions

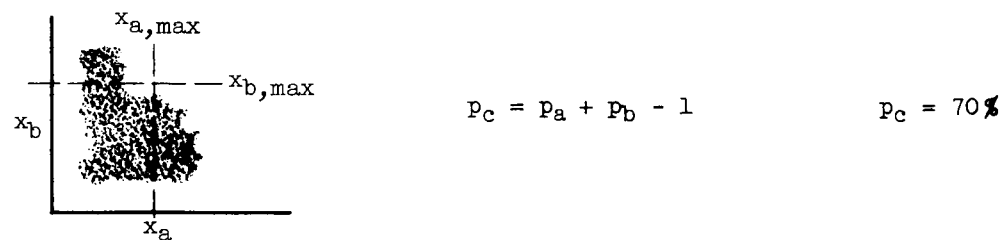


Combined probability if:

(1) Assumed preemptive:



(2) Assumed mutually exclusive:



(3) Assumed uncorrelated:

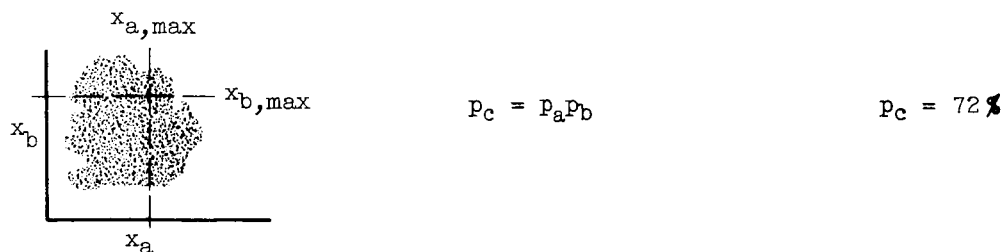


Figure 23. - Implications of various assumptions in combining probability distributions.